De nouveaux résultats sur la géométrie des mosaïques de Poisson-Voronoi et des mosaïques poissoniennes d'hyperplans. Etude du modèle de fissuration de Rényi-Widom

Pierre Calka

Université Claude Bernard Lyon I

De nouveaux résultats sur la géométrie des mosaïques de Poisson-Voronoi et des mosaïques poissoniennes d'hyperplans.

Etude du modèle de fissuration de Rényi-Widom.

Thèse soutenue le 5 décembre 2002
par
Pierre CALKA

en vue de l'obtention du
Diplôme de Doctorat
(arrêté du 30 mars 1992)
Spécialité : Mathématiques

Composition du jury :

François BACCELLI	Professeur (ENS)
André GOLDMAN	Professeur (Université Lyon 1), Directeur de thèse
Christian MAZZA	Professeur (Université Lyon 1)
Ilya MOLCHANOV	Professeur (Université de Berne, Suisse), Rapporteur
Didier PIAU	Professeur (Université Lyon 1)
Dietrich STOYAN	Professeur (Université de Freiberg, Allemagne), Rapporteur
Pierre VALLOIS	Professeur (Université Nancy 1)

194-2002

Remerciements

Je tiens tout d'abord à remercier André Goldman qui m'a initié à la géométrie aléatoire avec une passion communicative. Toujours attentif au cours de ces trois années, il m'a fait profiter de ses connaissances étendues et de sa grande imagination mathématique. Son intuition et ses encouragements m'ont apporté la détermination pour approfondir certaines pistes de recherche. Ce travail lui doit beaucoup, qu'il trouve ici le témoignage de ma profonde gratitude.

Ilya Molchanov et Dietrich Stoyan m'ont fait un grand honneur en acceptant d'être les rapporteurs de cette thèse. Je leur suis reconnaissant du soin qu'ils ont apporté à la lecture du texte et de toutes les remarques intéressantes dont ils m'ont fait part.

Je remercie Christian Mazza et Didier Piau pour leur disponibilité, l'intérêt qu'ils ont porté à mon travail et les discussions enrichissantes que j'ai eues avec chacun d'eux. Leur présence dans ce jury m'honore et me touche particulièrement.

François Baccelli a toute ma gratitude pour m'avoir fait l'honneur de faire partie de mon jury.

Je remercie par ailleurs chaleureusement Pierre Vallois et André Mézin pour m'avoir accordé leur confiance en me proposant de travailler avec eux sur le problème de la fissuration. Je suis très reconnaissant à Pierre Vallois de m'avoir permis à plusieurs reprises d'exposer nos travaux, ainsi que d'avoir accepté d'être dans ce jury.

Je tiens également à remercier tous les membres du laboratoire qui m'ont apporté leur aide pour des questions de mathématiques propres à la thèse, les enseignements, les problèmes informatiques ou autres. Je pense en particulier à Alexis, Anne, Eric, Frédérique, Gabriela, Jean, Jean-Baptiste, Mariam, Nadine, Nicolas, Pierre, Véronique et Madame Lefranc.

Merci enfin à mes proches dont le soutien a été essentiel tout au long de mes études.

Table des matières

Introduction.

La géométrie aléatoire remonte, comme la tradition le veut à l'expérience de l'aiguille de Buffon en 1777, au paradoxe de Bertrand sur les probabilités géométriques, et à la réplique fournie par H. Poincaré qui a conduit à la naissance de la géométrie intégrale puis de la géometrie stochastique (terme introduit par D. G. Kendall, K. Krickeberg et R. E. Miles en 1969). On pourra consulter par exemple les ouvrages classiques de H. Solomon [84], L. A. Santaló [77]. et G. Matheron [48]. De nos jours ce domaine a pris une telle extension, du fait notamment de ses innombrables implications dans les sciences expérimentales qu'il n'est pas possible d'en donner une vue exhaustive dans un volume raisonnable de lignes. Nous renvoyons pour cela aux excellents ouvrages de P. Hall [32], D. Stoyan et alt. [86], I. S. Molchanov [62], B. D. Ripley [76], etc.

Dans cette thèse, nous avons abordé quatre domaines de la géométrie stochastique.

A. Les mosaïques de Poisson-Voronoi ;
B. les mosaïques de Johnson-Mehl ;
C. les mosaïques poissoniennes d'hyperplans ;
D. le modèle de fissuration de Rényi-Widom.

A. Le principe de la construction de la mosaïque de Voronoi est le suivant. On se donne un sous-ensemble localement fini $\mathcal{P}$ d'un espace métrique sous-jacent (E, d), puis à tout élément $x \in \mathcal{P}$ (appelé *germe*), on associe la *cellule*

$$C(x) = \{y \in E; d(y, x) \leq d(y, x')\ \forall x' \in \mathcal{P}\},$$

constituée des points de E les plus proches de x. Dans le cas euclidien, on obtient ainsi une partition de l'espace en polyèdres convexes bordés par des portions d'hyperplans médiateurs des segments entre germes. Cette mosaïque fut introduite dans un cadre déterministe en dimension deux par G. L. Dirichlet [18] en 1850, puis en dimension supérieure par G. Voronoi [91] en 1908. Leur motivation était de résoudre des problèmes de minimisation de formes quadratiques prises sur des vecteurs à coordonnées entières.

En 1953, dans le but de modéliser la formation de cristaux, J. L. Meijering [50] a repris la construction de Voronoi avec un ensemble de germes aléatoires, plus exactement un processus ponctuel de Poisson. Le modèle obtenu, appelé la mosaïque de Poisson-Voronoi, est depuis largement utilisé pour modéliser différents phénomènes naturels. Selon R. Van de Weygaert [90], les galaxies sont situées à la périphérie de cellules de Poisson-Voronoi en modélisant les centres d'expansion de la matière par un processus de Poisson. De même, M. Gerstein, J. Tsai et M. Levitt [22] ont approché la forme d'une molécule de protéine en construisant les cellules de Voronoi associées à l'ensemble aléatoire des centres des atomes constituant cette molécule. Plus généralement, de nombreux problèmes concrets qui relèvent de territoires de prédominance se modélisent à l'aide de mosaïques de Poisson-Voronoi, notamment en météorologie [11], écologie [74] ou encore épidémiologie [49]. Citons enfin les applications aux télécommunications qui ont rencontré un grand succès récemment. On pourra consulter notamment les travaux de F. Baccelli et alt. [2], [3], [4], de S. G. Foss et S. A. Zuyev [21] ou I. S. Molchanov et S. A. Zuyev [61]. Les mosaïques de Voronoi aléatoires jouent également un rôle essentiel depuis quelques années en informatique dans les secteurs de la reconnaissance de formes [12] et de compression d'images [51].

Dans de nombreux cas, le modèle de Poisson-Voronoi apparaît naturellement comme le plus adapté. Il permet en outre d'utiliser certaines techniques propres au processus de Poisson. Cependant, il n'est pas le seul modèle possible de mosaïque de Voronoi aléatoire (voir par exemple l'étude générale conduite par J. Møller des mosaïques de Voronoi stationnaires [65] ou les travaux de G. Le Caër et J. S. Ho [44]). On pourra se reporter à ce sujet à l'ouvrage très complet de Okabe et alt. [69]. Dans notre travail, nous ne traiterons que le cas des mosaïques de Poisson-Voronoi.

Historiquement, les premiers travaux dans l'étude théorique de la mosaïque de Poisson-Voronoi remontent à 1961 et sont dûs à E. N. Gilbert [23], qui l'a utilisée pour modéliser la formation de cristaux et s'est intéressé aux propriétés de sa "cellule typique". Il a en particulier fourni le meilleur encadrement connu à ce jour de la queue de la loi du volume de la cellule typique. Depuis, R. E. Miles [57] en 1970 et J. Møller [63], [65] en 1994 ont précisé la notion de cellule typique. Ils ont de plus calculé les premiers moments des aires des faces k-dimensionnelles (k étant inférieur à la dimension de l'espace euclidien) et ont décrit la démarche qui permet d'obtenir les seconds moments ainsi que les corrélations s'exprimant par des formules assez compliquées. Tout récemment, A. Hayen et M. Quine ont donné une formule intégrale pour le troisième moment de l'aire en dimension deux [34]. Cependant, peu de distributions explicites ont été obtenus jusqu'à aujourd'hui, ce qui explique le recours intensif aux procédés de simulation (voir par exemple les résultats de A. L. Hinde et R. E. Miles [36] ou de S. Kumar et S. K. Kurtz [43]). Quelques résultats sur les lois ont été prouvés dernièrement dans le cadre de problèmes de télécommunications par S. G. Foss et S. A. Zuyev [21], S. A. Zuyev [94] et Baccelli et alt. [3]. Ceux-ci ont notamment déterminé un encadrement de la queue de la loi du rayon du disque circonscrit à la cellule typique en dimension deux et la loi, conditionnée par le nombre d'hyperfaces, du volume du "domaine fondamental". Par ailleurs, dans le cadre de la stéréologie, les propriétés des sections de la mosaïque de Poisson-Voronoi par des sous-espaces affines ont été étudiées successivement par L. Muche et D. Stoyan [67], S. N. Chiu, R. van de Weygaert et D. Stoyan [13] et L. Heinrich [35].

Enfin, on peut définir une mosaïque duale de la mosaïque de Voronoi, appelée mosaïque de Delaunay en reliant par des segments les germes dont les cellules associées ont des frontières communes. On obtient alors une partition de l'espace en simplexes. On connaît la loi explicite de la cellule typique de cette mosaïque [65] et on peut en déduire des résultats sur l'angle typique ou la distance typique entre deux germes voisins (voir l'article de L. Muche [66]).

Dans notre travail, nous exhibons dans le cas deux-dimensionnel les formules exactes de la loi du nombre de côtés de la cellule typique [9] et des lois conditionnelles, connaissant le nombre de côtés, de l'aire de la cellule, de son périmètre, de l'aire du domaine fondamental et plus généralement du vecteur constitué des positions relatives des côtés [10]. Notons que notre démarche s'applique également à la dimension supérieure, les formules qui en résultent étant néanmoins peu explicites.

Par ailleurs, toujours en dimension deux, nous précisons la loi conjointe du couple formé des rayons des disques (centrés à l'origine) inscrit et circonscrit de la cellule typique et en déduisons le caractère "circulaire" des cellules de la mosaïque ayant un grand rayon de disque inscrit [8]. La technique employée repose sur des propriétés de recouvrement du cercle qui ne s'étendent pas en dimension supérieure à deux.

Enfin, dans un travail en collaboration avec A. Goldman, nous nous sommes intéressés aux propriétés vibratoires des cellules de la mosaïque en toute dimension en étudiant le spectre du Laplacien avec condition de Dirichlet au bord de la cellule typique. Nous relions la fonction spectrale (ou transformée de Laplace de la fonction de répartition du spectre) à la trajectoire du pont brownien et en déduisons en dimension deux une estimation logarithmique de la loi de la première valeur propre [29], [30]. Le développement au voisinage de l'origine de l'espérance de la fonction spectrale fournit également des informations sur la géométrie des cellules.

B. Afin de représenter la formation de cristaux dans des systèmes métalliques générés par des germes en croissance, W. A. Johnson et R. F. Mehl [38] ont introduit en 1939 un modèle plus général que le précédent. En effet, si on place des germes au même instant dans l'espace $\mathbb{R}^d$ en les faisant croître ensuite à même vitesse, on obtient naturellement la mosaïque de Voronoi associée. En revanche, si ces germes apparaissent à des instants différents, la partition de l'espace obtenue n'est plus constituée de polyèdres convexes mais seulement de convexes étoilés bordés par des portions d'hyperboloïdes. La mosaïque obtenue est dite de Johnson-Mehl. Notons que le cas unidimensionnel présente également un intérêt non négligeable (voir par exemple le travail récent de S. N. Chiu et C. C. Yin [14]). Il sert en particulier de modèle stochastique pour la réplication de l'ADN [17].

L'étude mathématique de la mosaïque de Johnson-Mehl a été conduite par E. N. Gilbert [23] et J. Møller [64]. Théoriquement, il est possible de généraliser la plupart des calculs de moments obtenus dans le cas Poisson-Voronoi. Cependant, il devient rapidement ardu d'expliciter les intégrales en dimension supérieure à deux.

Dans cette thèse, nous avons cherché à prolonger l'étude du spectre du Laplacien des cellules de la mosaïque que nous avions menée dans le cas Poisson-Voronoi. De manière générale, on ne dispose pas d'informations précises sur les valeurs propres d'un domaine fixé sans propriété particulière (de convexité ou polyèdrale par exemple). Dans ce contexte, il est intéressant d'obtenir des résultats sur la fonction spectrale de la cellule typique de Johnson-Mehl. Précisément, nous relions celle-ci au pont brownien et déterminons un développement à deux termes au voisinage de l'origine via des conditions explicites d'intégrabilité [29].

C. C'est dans le cadre physique très précis de l'étude des trajectoires de particules dans les chambres à bulles que S. A. Goudmit [31] a introduit en 1945 la mosaïque poissonienne d'hyperplans en dimension deux. En 1964, R. E. Miles utilise cette mosaïque afin de modéliser les fibres de papier. Il a fourni à cette occasion l'essentiel des résultats en loi et des calculs de moments empiriques connus aujourd'hui [54], [55]. De manière générale, hormis quelques exceptions notables comme les travaux ultérieurs de R. E. Miles [56], [58] et ceux de G. Matheron [48], la plupart des écrits sur la mosaïque poissonienne d'hyperplans se limitent au cas deux-dimensionnel. Récemment, A. Goldman a étudié le spectre du Laplacien des cellules [26] et a par ailleurs [28] précisé une ancienne conjecture dûe à D. G. Kendall (voir par exemple [86]) selon laquelle les cellules dont l'aire est "grande" ont une forme approximativement circulaire. Ce dernier travail a été prolongé depuis par I. N. Kovalenko [42]. Ajoutons que K. Paroux a obtenu des théorèmes centraux-limites dans ce contexte [72].

Dans notre travail, nous donnons des formules explicites en dimension deux des lois du nombre de côtés de la cellule typique (resp. de la cellule contenant l'origine) et des lois conditionnelles, connaissant le nombre de côtés, de l'aire de la cellule, de son périmètre et du vecteur constitué des positions des côtés de la cellule typique (resp. de la cellule contenant l'origine) [10]. Comme dans le cas des mosaïques de Poisson-Voronoi, la technique se généralise en toute dimension bien qu'il soit plus difficile d'expliciter les intégrales.

De plus, toujours en dimension deux, nous calculons la loi conjointe du couple formé des rayons des disques (centrés à l'origine) inscrit et circonscrit de la cellule contenant l'origine. Nous en déduisons une nouvelle vérification théorique de la conjecture de D. G. Kendall [8].

Par ailleurs, nous prolongeons à toute dimension un résultat dû à R. E. Miles en dimension deux [59] en exhibant une réalisation explicite de la cellule typique à partir de sa boule inscrite et de son simplexe circonscrit dont nous donnons les lois exactes [6]. Enfin, nous retrouvons de manière rigoureuse en toute dimension un résultat de R. E. Miles [54], énoncé en dimension deux, concernant les mosaïques poissoniennes d'hyperplans épaissies [7].

D. On considère un système physique constitué d'un substrat recouvert d'un dépôt d'épaisseur négligeable comme, par exemple, une couche de terre sur un étang ou une couche de peinture sur un mur. Lorsque l'on applique une force unidirectionnelle à cet ensemble, on constate expérimentalement du fait de son inhomogénéité, la formation de fissures (sur le plan du dépôt) que l'on modélise en première approximation par des droites parallèles entre elles. Etudier leur position revient à décrire l'ensemble constitué des projections de ces droites sur un axe dirigé par la force de traction. La représentation probabiliste de ce phénomène doit prendre en compte le principe physique de relaxation de la contrainte selon lequel aucune fissure ne peut apparaître "à proximité" d'une fissure préexistante. C'est pourquoi il est licite de choisir le modèle de Rényi-Widom, connu également dans la littérature sous le nom de modèle "des places de parkings". Plus précisément, on dispose d'une suite ordonnée de points que l'on appelle les "positions potentielles", indépendants et identiquement distribués de loi uniforme sur un intervalle fixé. Par ailleurs, on fixe $r > 0$. On conserve le premier point, puis de manière récursive, on décide de conserver le $n^{\text{ième}}$ point, $n \geq 1$, lorsque sa distance aux points conservés jusqu'à l'étape $(n-1)$ est supérieure à r. Dans le cas contraire, on le supprime et on poursuit l'algorithme. Dans le cadre de notre modèle, chaque point représente la position d'une fissure.

Il est *a priori* difficile de déterminer la loi (ou même le premier moment) du nombre de fissures effectivement construites sur l'intervalle lorsqu'on dispose d'un nombre fixe (ou poissonien) de fissures "potentielles" au départ. Ainsi, les principaux résultats sur ce sujet sont de nature asymptotique lorsque l'on fait tendre la longueur de l'intervalle vers l'infini. En 1958, A. Rényi [75] a étudié le modèle à saturation, c'est-à-dire lorsqu'il n'est plus possible de rajouter de nouveau point sur l'intervalle. Il a obtenu en particulier une formule explicite de l'intensité limite (nombre moyen de points par unité de longueur). B. Widom [92] a par la suite calculé par des méthodes plus ou moins heuristiques, la loi limite d'une "distance typique" entre deux points successifs lorsque le nombre de points "potentiels" est proportionnel à la longueur de l'intervalle. Il paraît donc naturel de chercher à obtenir un processus ponctuel limite en loi sur la droite réelle lorsqu'on étend les bornes de l'intervalle de départ à l'infini.

Dans un travail en collaboration avec A. Mézin et P. Vallois, nous définissons directement l'objet limite comme étant un processus ponctuel stationnaire et ergodique sur la droite en associant à chaque position de fissure une seconde coordonnée qui correspond au niveau de contrainte exact auquel elle apparaît. Nous procédons à l'étude statistique de ce modèle en déterminant explicitement l'intensité de fissuration et la loi conjointe d'un couple constitué de la "distance inter-fissures typique" et du "niveau de contrainte typique associé". Nous retrouvons en particulier les résultats de A. Rényi et B. Widom. Nous revenons ensuite à une vision "locale" du processus point par point en déterminant la loi conjointe des n premières positions de fissures à droite de l'origine, $n \geq 1$. Nous montrons en particulier qu'il s'agit de la loi des n premiers points d'un processus de renouvellement conditionné et nous en déduisons un algorithme de simulation.

D'autres mosaïques aléatoires ont été étudiées dans la littérature. On peut tout d'abord citer les nombreuses généralisations des mosaïques de Voronoi aléatoires [69], en considérant notamment les n plus proches voisins, $n \geq 2$ (voir [57] pour plus de détails), ou en prenant une autre métrique que la métrique euclidienne (voir [37] pour une étude des cellules de Voronoi sur des espaces hyperboliques). On peut aussi songer à des modèles où les arêtes et non plus les germes croissent avec le temps (voir notamment le travail de E. N. Gilbert [24]). Plus récemment, M. Schlather et D. Stoyan [78] ont mené l'étude d'un modèle de fissuration bidimensionnel fourni par des mosaïques de Voronoi partiellement effacées.

Chapitre 1

La cellule typique comme moyen d'étude statistique d'une mosaïque aléatoire stationnaire.

1.1 Introduction.

Dans ce travail, trois types de mosaïques aléatoires de l'espace euclidien $\mathbb{R}^d$, $d \geq 2$, sont considérés :

(i) la mosaïque de Poisson-Voronoi $\mathcal{V}_d$;

(ii) la mosaïque de Johnson-Mehl $\mathcal{J}_d$;

(iii) la mosaïque poissonienne d'hyperplans $\mathcal{P}_d$.

(i) Rappelons que la mosaïque de Poisson-Voronoi $\mathcal{V}_d$ [50], [23], s'obtient en prenant un processus ponctuel de Poisson Φ sur $\mathbb{R}^d$ de mesure d'intensité la mesure de Lebesgue canonique notée V_d et en considérant la partition de l'espace $\mathbb{R}^d$ constitué par les cellules

$$C(x) = \{y \in \mathbb{R}^d; ||y - x|| \leq ||y - x'||, x' \in \Phi\}, \quad x \in \Phi.$$

Notons que pour tout $x \in \Phi$, l'ensemble $C(x)$ est un polyèdre convexe borné.

(ii) Plus généralement, soit $\Phi = \{(x_i, t_i) \in \mathbb{R}^d \times \mathbb{R}_+, i \geq 1\}$ un processus ponctuel de Poisson dans l'espace $\mathbb{R}^d \times \mathbb{R}_+$ de mesure d'intensité $V_d(dx)\Lambda(dt)$, où Λ est une mesure localement finie sur $\mathbb{R}_+$ satisfaisant les conditions "canoniques" imposées par J. Møller [64] :

$$\Lambda([0, \infty)) > 0, \tag{1.1}$$

et

$$\lambda = \int p(t)\Lambda(dt) < +\infty \tag{1.2}$$

où

$$p(t) = \exp\left(-\omega_d \int_0^t (t - w)^d \Lambda(dw)\right), \tag{1.3}$$

ω_d étant le volume de la boule-unité de $\mathbb{R}^d$.

On construit la mosaïque de Johnson-Mehl de la manière suivante : pour tout $(x, t) \in \Phi$, un germe naît à la position x dans l'espace $\mathbb{R}^d$ à l'instant $t \geq 0$. Puis il grossit à vitesse constante (prise égale à 1) de telle manière qu'il atteint le point $y \in \mathbb{R}^d$ à l'instant $T_{(x,t)}(y) = t + ||x - y||$.

On définit la cellule $C(x, t)$ associée au couple (x, t) comme l'ensemble

$$C(x, t) = \{y \in \mathbb{R}^d; T_{(x,t)}(y) \leq T_{(x',t')}(y), (x', t') \in \Phi\}.$$

L'ensemble de toutes les cellules non vides constitue la *mosaïque de Johnson-Mehl* $\mathcal{J}_d$ [23].

Lorsque Λ est une mesure de Dirac, c'est-à-dire que tous les germes apparaissent simultanément dans $\mathbb{R}^d$, $\mathcal{J}_d$ est une mosaïque de Poisson-Voronoi. Cependant, dans le cas contraire, les cellules sont des domaines bornés par des portions d'hyperboloïdes.

Remarquons par ailleurs que la condition (1.2) garantit que le nombre de cellules non vides dont le germe associé appartient à un borélien fixé de mesure de Lebesgue finie, est d'espérance finie [64].

(iii) Finalement, soit Φ un processus de Poisson dans $\mathbb{R}^d$ de mesure d'intensité

$$\mu(A) = \mathbf{E} \sum_{x \in \Phi} \mathbf{1}_A(x) = \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r, u) d\nu_d(u) dr, \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{1.4}$$

Pour tout $x \in \mathbb{R}^d$, on considère l'hyperplan polaire $H(x)$ associé défini par

$$H(x) = \{y \in \mathbb{R}^d; (y - x) \cdot x = 0\}, \tag{1.5}$$

où $\cdot$ est le produit scalaire usuel de $\mathbb{R}^d$.

L'ensemble

$$\mathcal{H}(\Phi) = \{H(x_i); x_i \in \Phi\}$$

est appelé un *processus poissonien d'hyperplans* dans $\mathbb{R}^d$ et la fermeture d'une composante connexe de l'ensemble

$$\mathbb{R}^d \setminus \cup_{x_i \in \Phi} H(x_i)$$

une *cellule* associée à $\mathcal{H}$. Alors presque sûrement, les cellules sont des polyèdres convexes qui constituent une partition de $\mathbb{R}^d$, appelée la *mosaïque poissonienne d'hyperplans* $\mathcal{P}_d$ [54], [55].

Ces trois types de mosaïques aléatoires sont stationnaires. Pour d'autres exemples de mosaïques stationnaires, on pourra consulter [86] ou [69].

Afin de conduire l'étude statistique des mosaïques aléatoires stationnaires (en vue des nombreuses applications, voir les deux ouvrages déjà cités), on introduit classiquement la notion de cellule typique. De fait, celle-ci a été définie de différentes manières suivant les modèles et les écoles mathématiques.

Ainsi, historiquement une première approche remonte au travail de C. Palm [70] dans le domaine des télécommunications et consiste à associer à chaque cellule C un point $z(C)$ (par exemple le centre de la boule inscrite dans C, son germe pour les mosaïques $\mathcal{V}_d$ et $\mathcal{J}_d$, etc) de telle manière que l'ensemble des points choisis constitue un processus stationnaire Ψ d'intensité finie ψ. On définit ensuite la cellule typique $\mathcal{C}$ en loi : pour toute application mesurable bornée h définie sur l'ensemble des connexes compacts de $\mathbb{R}^d$,

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\psi V_d(B)} \mathbf{E} \sum_{x \in \Psi; x = z(C)} h(C(x) - x),$$

où B est un ensemble borélien fixé de $\mathbb{R}^d$, vérifiant $0 < V_d(B) < +\infty$..

Cette définition ne dépend pas du borélien B choisi. La technique est usuellement appliquée pour l'introduction des mosaïques de Poisson-Voronoi [63],[65] et de Johnson-Mehl [64]. Intuitivement, ce procédé revient à “choisir une cellule au hasard” dans la mosaïque aléatoire.

Une seconde approche consiste à appliquer la technique empirique (l'échantillonage). On considère une fonctionnelle h sur les connexes compacts de $\mathbb{R}^d$ invariante par translation et on considère la moyenne empirique des $h(C)$ prise sur toutes les cellules C qui interceptent la boule centrée en l'origine, de rayon $R \geq 0$. On prouve que ces moyennes empiriques convergent quand R tend vers l'infini, vers une constante presque-sûre, appelée la moyenne empirique de h,

Une troisième approche consiste à étudier une cellule particulière (par exemple la cellule C_0 contenant l'origine) en la renormalisant convenablement. La cellule C_0 s'appelle la *cellule de Crofton* dans le cas de la mosaïque $\mathcal{P}_d$.

Contrairement au cas des mosaïques de Poisson-Voronoi et de Johnson-Mehl, l'étude statistique des mosaïques poissoniennes d'hyperplans a été conduite par R. E. Miles [54], [55], [59] (et son école [16], [48]) au moyen de la cellule empirique.

De fait, en ce qui concerne les mosaïques $\mathcal{V}_d$ et $\mathcal{J}_d$, ces trois notions coïncident et pour ce qui est de la mosaïque $\mathcal{P}_d$, il est possible de construire dans ce cas-là une cellule typique par la méthode de Palm qui coïncide également avec la cellule empirique de Miles comme avec la cellule contenant l'origine convenablement renoramalisée.

Le fait que la cellule typique puisse être vue de différentes manières permet d'avoir plusieurs angles d'attaque possibles de problèmes ouverts. Il importe donc de prouver rigoureusement l'identité entre ces trois notions.

Concernant la mosaïque $\mathcal{V}_d$, l'égalité entre la cellule typique et la cellule C_0 renormalisée a été prouvée par J. Møller ([65], Prop. 3.3.2). Par ailleurs, il est facile de voir via le théorème ergodique de Wiener que les moyennes empiriques existent et coïncident avec les moyennes prises sur C_0 renormalisée. L'application du théorème ergodique est triviale. Cependant, une réelle difficulté apparaît dans le traitement des éléments de la mosaïque qui coupent le bord de la boule de rayon $R \geq 0$. Cette question a été examinée par R. Cowan en dimension deux [16], [15]. Le passage à la dimension supérieure présente quelques difficultés supplémentaires que nous traitons.

Pour la mosaïque $\mathcal{J}_d$, l'égalité entre la cellule typique et la cellule C_0 renormalisée n'apparaît pas explicitement dans la littérature mais s'obtient facilement en reprenant mot pour mot la méthode utilisée par Møller pour $\mathcal{V}_d$. La cellule empirique n'a pas été jusquà présent étudiée. Nous montrons son existence et l'égalité avec C_0 renormalísee moyennant une hypothèse d'intégrabilité portant sur la mesure d'intensité temporelle.

Concernant la mosaïque $\mathcal{P}_d$, l'existence de la cellule empirique et son égalité avec la cellule C_0 renormalisée (c'est-'a-dire le fait que les “effets de bord” sont négligeables) a été également traité en dimension deux par R. Cowan [16], [15], et K. Paroux [71]. Le traitement que nous utilisons pour $\mathcal{V}_d$, $d \geq 3$, s'applique via quelques modifications mineures à la mosaïque $\mathcal{P}_d$. L'idée que la cellule empirique puisse être construite par la méthode de Palm est nouvelle et nous permettra par la suite d'obtenir des résultats originaux.

Un des outils essentiels dans l'étude des mosaïques $\mathcal{V}_d$ et $\mathcal{J}_d$ est la formule de Slivnyak [83]. Cette formule est en particulier très utile pour réaliser la cellule typique. Dans le cas de la mosaïque $\mathcal{P}_d$, le fait que la cellule typique puisse être construite via un procédé de Palm mous permettra par la suite (voir chapitre 4) d'appliquer la formule de Slivnyak.

1.2 La formule de Slivnyak.

Soit Φ un processus ponctuel de Poisson dans $\mathbb{R}^d$ de mesure d'intensité

$$M_I(A) = \mathbf{E}\sum_{x\in\Phi}\mathbf{1}_A(x).$$

Le processus Φ est une application mesurable à valeurs dans l'espace $\mathcal{M}_\sigma$ des suites localement finies, ce dernier étant muni de la tribu cylindrique $\mathcal{T}_c$ engendrée par les applications

$$\varphi_A : \left\{ \begin{array}{lcl} \mathcal{M}_\sigma & \longrightarrow & \mathbb{N}\cup\{+\infty\} \\ \gamma & \longmapsto & \#(A\cap\gamma) \end{array} \right. , \quad A\in\mathcal{B}(\mathbb{R}^d).$$

Pour toute fonction f positive mesurable définie sur l'espace produit $(\mathbb{R}^d)^n\times\mathcal{M}_\sigma$, $n\in\mathbb{N}^*$, et invariante par permutation des n premières coordonnées, on dispose de la formule de Slivnyak (voir par exemple [65]) :

$$\mathbf{E}\sum_{\xi\in\overline{\Phi}^{(n)}} f(\xi,\Phi) = \frac{1}{n!}\int \mathbf{E}f\left(\xi,\Phi\cup\xi\right)d\overline{M_I}^{(n)}(\xi), \tag{1.6}$$

où $\overline{\Phi}^{(n)}$ désigne l'ensemble des parties de Φ de cardinal n, et où

$$d\overline{M_I}^{(n)}(\xi) = dM_I(\xi_1)\cdots dM_I(\xi_n), \quad \xi=\{\xi_1,\cdots\xi_n\}\in\overline{\Phi}^{(n)}.$$

1.3 La mosaïque de Poisson-Voronoi.

1.3.1 La cellule typique au sens de Palm.

Introduisons la notion de *cellule typique* $\mathcal{C}$ au sens de Palm. On considère pour cela l'espace $\mathcal{K}$ des ensembles convexes compacts de $\mathbb{R}^d$ muni de la topologie usuelle de Hausdorff. Fixons un ensemble borélien borné $B\subset\mathbb{R}^d$ de mesure de Lebesgue non nulle. Classiquement, la cellule typique $\mathcal{C}$ est définie par l'identité [65] :

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_d(B)}\mathbf{E}\sum_{x\in B\cap\Phi} h(C(x)-x),$$

où $h:\mathcal{K}\longrightarrow\mathbb{R}$ parcourt l'ensemble des fonctions numériques, mesurables et bornées.

Considérons par ailleurs la cellule

$$C(0) = \{y\in\mathbb{R}^d; ||y||\le||y-x||, x\in\Phi\},$$

obtenue en rajoutant l'origine au processus ponctuel Φ. On démontre alors en utilisant la formule de Slivnyak (voir [65]) que $C(0)$ coïncide en loi avec la cellule typique $\mathcal{C}$.

Par ailleurs, il est bien connu que $\mathcal{C}$ est reliée en loi à la cellule contenant l'origine C_0 de la manière suivante ([65], Prop. 3.3.2.) :

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}(1/V_d(C_0))}\mathbf{E}\left\{\frac{h(C_0)}{V_d(C_0)}\right\}, \tag{1.7}$$

pour toute fonction mesurable bornée $h : \mathcal{K} \longrightarrow \mathbb{R}$ invariante par translation.

1.3.2 Convergence des moyennes empiriques.

Montrons à présent la convergence des moyennes empiriques sur la mosaïque $\mathcal{V}_d$. En dimension $d \geq 3$, cette propriété est nouvelle. La preuve repose sur le théorème ergodique de Wiener [93]. Aussi, nous allons tout d'abord démontrer l'ergodicité de la mosaïque .

On réalise Ω comme l'espace $\mathcal{M}_\sigma$ muni de la tribu $\mathcal{T}_c$. Φ est alors l'application identité sur Ω.

Pour tout $a \in \mathbb{R}^d$ fixé, l'application

$$\{x_i; i \geq 1\} \longmapsto \{x_i + a; i \geq 1\}$$

induit classiquement une transformation $T^a : \Omega \longrightarrow \Omega$ préservant la mesure.

Proposition 1.3.1 *T^a, $a \in \mathbb{R}^d$, préserve la mesure et est ergodique.*

Preuve. L'invariance par translation de la mesure de Lebesgue sur $\mathbb{R}^d$ implique que T^a preserve la probabilité $\mathbf{P}$.

Pour prouver l'ergodicité, il suffit de vérifier que T^a est fortement mélangeante, c'est-à-dire que pour tous A, B boréliens bornés de $\mathbb{R}^d$ et tous $k, l \in \mathbb{N}$,

$$\lim_{|a|\to+\infty} \mathbf{P}\{\#(\Phi\cap A) = k) \cap (\#(T^{-a}(\Phi)\cap B) = l)\} = \mathbf{P}\{\#(\Phi\cap A) = k\} \cdot \mathbf{P}\{\#(\Phi\cap B) = l\}. \tag{1.8}$$

Remarquons $\#(T^{-a}(\Phi) \cap B) = \#(\Phi \cap (B + a))$ et que les ensembles A et $B + a$ sont disjoints pour $|a|$ suffisamment grand. Par conséquent, les évènements $\{\#(\Phi \cap A) = k\}$ et $\{\#(T^{-a}(\Phi) \cap B) = l\}$ sont indépendants. Nous avons donc

$$\begin{aligned}\mathbf{P}\{(\#(\Phi \cap A) = k) \cap (\#(T^{-a}(\Phi) \cap B) = l)\} \\ &= \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap (B + a)) = l\} \\ &= \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(T^{-a}(\Phi) \cap B) = l\} \\ &= \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\},\end{aligned}$$

la dernière égalité se déduisant du fait que Φ est invariant par T^{-a}.

On a donc bien prouvé l'égalité (1.8), ce qui implique l'ergodicité de la mesure .

□

Notons $\mathcal{C}_R$ (resp. $\mathcal{C}'_R$) l'ensemble des cellules de $\mathcal{V}_d$ incluses dans la boule ouverte $B(R)$ centrée à l'origine et de rayon $R > 0$ (resp. interceptant la frontière de $B(R)$) et $N_R = \#\mathcal{C}_R$ (resp. $N'_R = \#\mathcal{C}'_R$).

Lemme 1.3.2 *$(N_R + N'_R)$ est intégrable.*

Preuve. Soit L_0 le rayon du plus petit disque centré à l'origine qui n'intercepte pas Φ. La loi de L_0 est alors fournie par l'égalité

$$\mathbf{P}\{L_0 \geq r\} = e^{-\omega_d r^d}, \quad r \geq 0,$$

où ω_d est la mesure de Lebesgue de la boule-unité de $\mathbb{R}^d$.

Il est bien connu que conditionnellement à l'évènement $\{L_0 = r\}$, $r > 0$, Φ est égal en loi à $\Phi_r \cup \{X_0\}$, où Φ_r est un processus de Poisson de mesure d'intensité $\mathbf{1}_{B(r)^c} \cdot V_d$ et X_0 est un point indépendant uniformément distribué sur la sphère $\partial B(r)$.

Soit $x \in \Phi$ tel que $C(x) \cap B(R) \neq \emptyset$, c'est-à-dire qu'il existe un point $y \in B(R)$ tel que $||y - x|| \leq ||y - X_0||$. Nous avons alors $||x|| \leq 2R + L_0$, et donc

$$N_R + N'_R \leq \#(\Phi \cap D(2R + L_0)). \tag{1.9}$$

En prenant l'espérance de l'égalité (1.9), nous déduisons que

$$\begin{aligned} \mathbf{E}(N_R + N'_R) &\leq \int_0^{+\infty} \mathbf{E}\{\#(\Phi \cap D(2R + L_0))|L_0 = r\} d\omega_d r^{d-1} e^{-\omega_d r^d} dr \\ &\leq \int_0^{+\infty} (\omega_d((2R + r)^d - r^d) + 1) d\omega_d r^{d-1} e^{-\omega_d r^d} dr < +\infty. \end{aligned}$$

□

Considérons $h : \mathcal{K} \longrightarrow \mathbb{R}_+$ une fonction mesurable, invariante par translation et bornée (K_h désignant une borne de $|h|$). En appliquant les égalités $C_0(T^{-a}\omega) = C_a(\omega) - a$, $a \in \mathbb{R}^d$, $\omega \in \Omega$, nous obtenons les identités presque-sûres suivantes :

$$\frac{1}{v(R)} \int_{B(R)} \frac{dx}{V_d(C_0(T^{-x}\omega))} = \frac{N_R(\omega)}{v(R)} + \frac{1}{v(R)} \varepsilon(R, \mathbf{1}, \omega) \tag{1.10}$$

$$\frac{1}{v(R)} \int_{B(R)} \frac{h(C_0(T^{-x}\omega))}{V_d(C_0(T^{-x}\omega))} dx = \frac{1}{v(R)} \sum_{C \in \mathcal{C}_R} h(C_i(\omega)) + \frac{1}{v(R)} \varepsilon(R, h, \omega), \tag{1.11}$$

où $v(R) = V_d(B(R))$, et

$$\varepsilon(R, h, \cdot) = \sum_{C \in \mathcal{C}'_R} h(C) \cdot \frac{V_d(C \cap B(R))}{V_d(C)} \text{ p.s..}$$

En prenant l'espérance de (1.11), nous obtenons que

$$\mathbf{E}\left(\frac{|h(C_0)|}{V_d(C_0)}\right) \leq \frac{K_h}{v(R)} \mathbf{E}(N_R + N'_R) < +\infty.$$

Comme les transformations T^a, $a \in \mathbb{R}^d$, sont ergodiques, nous pouvons appliquer le théorème ergodique de Wiener [93]. Par conséquent, en supposant que lorsque $R \to +\infty$, la contribution du reste $\varepsilon(R, h, \omega)$ disparaît, c'est-à-dire

$$\varepsilon(R, h, \cdot)/v(R) \to 0 \text{ p.s. quand } R \to +\infty, \tag{1.12}$$

nous disposons du théorème suivant.

Théorème 1.3.3 *Pour toute application h mesurable, bornée et invariante par translation, presque sûrement,*

$$\widetilde{\mathbf{E}}h = \lim_{R \longrightarrow \infty} \frac{1}{N_R} \sum_{C \in \mathcal{C}_R} h(C) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right). \tag{1.13}$$

En particulier, $\widetilde{\mathbf{E}}h = \mathbf{E}h(\mathcal{C})$.

Afin de prouver le théorème 1.3.3, il faut vérifier que la technique est valable, c'est-à-dire que la convergence (1.12) est vraie. Nous avons p.s. l'inégalité

$$\frac{|\varepsilon(R, h, \cdot)|}{v(R)} \leq K_h \cdot \frac{N'_R}{v(R)}.$$

Par conséquent, il suffit de montrer la proposition suivante qui est la difficulté principale dans la construction des lois empiriques :

Proposition 1.3.4 *Quand $R \to +\infty$, on a :*

$$N'_R / v(R) \to 0 \text{ p.s.}.$$

Preuve. Nous proposons ici une généralisation à toute dimension de l'argument fourni par R. Cowan dans le cas deux-dimensionnel [15], [16]. Afin de prouver la proposition 1.3.4, nous devons introduire quelques nouvelles notations. Considérons pour tout $0 \leq k \leq d$:

(i) $N_{R,k}$ le nombre de faces de dimension k de $\mathcal{V}_d$ incluses dans $B(R)$;

(ii) $N'_{R,k}$ le nombre de faces de dimension k interceptant la frontière de $B(R)$;

(iii) $S_{R,k}$ la mesure de Hausdorff k-dimensionnelle de l'intersection de $B(R)$ avec les faces k-dimensionnelles de $\mathcal{V}_d$.

On pourra se reporter à [65] pour une définition des faces k-dimensionnelles de $\mathcal{V}_d$. En particulier, remarquons que $N_R = N_{R,d}$ (resp. $N'_R = N'_{R,d}$). Nous allons prouver que toutes ces variables sont intégrables. Puisque toute k-face de la mosaïque est l'intersection d'exactement $(d + 1 - k)$ cellules, $0 \leq k \leq d$, nous avons

$$(N_{R,k} + N'_{R,k}) \leq \binom{N_R + N'_R}{d + 1 - k} \leq \frac{(N_R + N'_R)^{d+1-k}}{(d + 1 - k)!} \text{ p.s.}. \tag{1.14}$$

La même technique que dans la preuve du lemme 1.3.2 permet de montrer que $\mathbf{E}\{(N_R + N'_R)^{d+1}\} < +\infty$. Ainsi, nous déduisons de (1.14) que $\mathbf{E}(N_{R,k} + N'_{R,k}) < +\infty$, $0 \leq k \leq d$.

Par ailleurs, nous avons clairement

$$S_{R,k} \leq \omega_k R^k (N_{R,k} + N'_{R,k}) \text{ p.s.},$$

donc $S_{R,k}$ est également intégrable pour tout $0 \leq k \leq d$.

Afin de prouver la proposition 1.3.4, nous avons besoin de deux lemmes intermédiaires :

Lemme 1.3.5 *Pour tout $0 \leq k \leq d$, $S_{R,k}/v(R)$ converge p.s. vers une constante lorsque R tend vers l'infini.*

Lemme 1.3.6 *Pour tout $0 \leq k \leq d$, $N'_{R,k}/v(R)$ converge vers 0 p.s. lorsque R tend vers l'infini.*

En appliquant le lemme 1.3.6 à $k = d$, nous obtenons clairement la proposition 1.3.4.

□

Concentrons-nous à présent sur la preuve des deux lemmes :
Preuve du lemme 1.3.5. Fixons $0 \leq k \leq d$ et montrons le résultat intermédiaire suivant :

$$\int_{B(R-y)} S_{y,k}(T^{-t}(\omega))dt \leq v(y)S_{R,k}(\omega) \leq \int_{B(R+y)} S_{k,y}(T^{-t}(\omega))dt \text{ p.s., } 0 < y < R. \quad (1.15)$$

En effet, en désignant par $\mathcal{F}_{k,R}$ l'ensemble des k-faces de la mosaïque interceptant $B(R)$ et par $\mu_{k,F}$ la mesure de Hausdorff k-dimensionnelle de la face F, $F \in \mathcal{F}_{k,R}$, nous avons

$$\begin{aligned}
\int_{B(R-y)} S_{y,k}(T^{-t}(\omega))dt &= \int \mathbf{1}_{B(R-y)}(t) \sum_{F \in \mathcal{F}_{k,R}(\omega)} \int \mathbf{1}_{B(y)}(s)\mathbf{1}_F(s+t)d\mu_{k,F}(s+t)dt \\
&= \sum_{F \in \mathcal{F}_{k,R}(\omega)} \int \mathbf{1}_{B(R-y)}(t) \int \mathbf{1}_{B(y)}(u-t)\mathbf{1}_F(u)d\mu_{k,F}(u)dt \\
&\leq \sum_{F \in \mathcal{F}_{k,R}(\omega)} \int \left[\int \mathbf{1}_{B(y)}(u-t)dt\right] \mathbf{1}_{B(R)}(u)\mathbf{1}_F(u)d\mu_{k,F}(u) \\
&\leq v(y) \cdot S_{R,k}(\omega) \quad \text{p.s.,}
\end{aligned}$$

ce qui prouve la partie droite de l'encadrement (1.15).

Par ailleurs, nous avons également

$$\begin{aligned}
\int_{B(R+y)} S_{y,k}(T^{-t}(\omega))dt &= \sum_{F \in \mathcal{F}_{k,R+2y}(\omega)} \int \left[\int \mathbf{1}_{B(R+y)}(t)\mathbf{1}_{B(y)}(u-t)dt\right] \mathbf{1}_F(u)d\mu_{k,F}(u) \\
&\geq \sum_{F \in \mathcal{F}_{k,R}(\omega)} \int \left[\int \mathbf{1}_{B(y)}(u-t)dt\right] \mathbf{1}_{B(R)}(u)\mathbf{1}_F(u)d\mu_{k,F}(u) \\
&\geq v(y) \cdot S_{R,k}(\omega) \quad \text{p.s.,}
\end{aligned}$$

ce qui conclut la démonstration du résultat (1.15).

Revenons à présent à la preuve du lemme 1.3.5. L'encadrement (1.15) implique presque sûrement que

$$\frac{v(R-y)}{v(y)v(R)} \int_{B(R-y)} \frac{S_{y,k}(T^{-t}(\omega))}{v(R-y)}dt \leq \frac{S_{R,k}(\omega)}{v(R)} \leq \frac{v(R+y)}{v(y)v(R)} \int_{B(R+y)} \frac{S_{y,k}(T^{-t}(\omega))}{v(R+y)}dt.$$

D'après le théorème ergodique de Wiener, les expressions à gauche et à droite tendent vers
$(\mathbf{E}(S_{y,k})/v(y))$ p.s.. Ainsi, on en déduit la convergence presque-sûre de $S_{R,k}/v(R)$.

$\square$

Preuve du lemme 1.3.6. Nous allons montrer la convergence

$$(C)_k : \quad N'_{R,k}/v(R) \to 0 \text{ p.s. quand } R \to +\infty,$$

en utilisant un raisonnement par récurrence sur $k \in [0, d]$.

Puisque $N'_{R,0} = 0$ p.s., $(C)_0$ est bien vérifiée. Supposons à présent que la convergence $(C)_{k-1}$ est satisfaite pour un k fixé, $1 \leq k \leq d-1$ et montrons $(C)_k$.

Considérons pour ce faire un réel $y \in (0, R)$ fixé. Si une k-face intercepte la frontière de $B(R)$ de telle manière que l'intersection de ses sous-$(k-1)$-faces avec l'ensemble $B(R+y) \setminus B(R)$ est vide, alors la mesure k-dimensionnelle de l'intersection de cette k-face avec $B(R+y) \setminus B(R)$ est au moins égale au volume d'une boule k-dimensionnelle de rayon $\sqrt{(R+y)^2 - R^2}$, c'est-à-dire

$$A_k = \omega_k((R+y)^2 - R^2)^{k/2}.$$

Par conséquent, en notant L_k le nombre de tels k-faces, nous avons

$$S_{R+y,k} - S_{R,k} \geq A_k \cdot L_k, \tag{1.16}$$

De plus, nous obtenons aisément la formule

$$L_k = N'_{R,k} - \#\{k\text{-faces ayant une de ses } (k-1)\text{-faces qui intercepte } B(R+y) \setminus B(R)\}. \tag{1.17}$$

Par ailleurs, pour tout $0 \leq l \leq l' \leq d$, une face l-dimensionnelle de $\mathcal{V}_d$ est incluse dans exactement $\binom{d-l+1}{l'-l}$ différentes faces l'-dimensionnelles (voir par exemple [65], Prop. 2.1.1.)).

En particulier, une $(k-1)$-face fixée est incluse dans exactement $(d-k+2)$ k-faces. Ce fait associé à (1.17) a pour conséquence que

$$L_k \quad \geq N'_{R,k} - (d-k+2)\#\{(k-1)\text{-faces interceptant } B(R+y) \setminus B(R)\}. \tag{1.18}$$

Il reste à voir que toute $(k-1)$-face interceptant $B(R+y) \setminus B(R)$ peut :
- soit intercepter la frontière de $B(R+y)$;
- soit intercepter la frontière de $B(R)$;
- soit être incluse dans $B(R+y) \setminus B(R)$.

Dans le dernier cas, nous remarquons que la $(k-1)$-face a alors nécessairement une sous 0-face incluse dans $B(R+y) \setminus B(R)$. Ainsi, nous obtenons que

$$\begin{aligned}
&\#\{(k-1)\text{-faces interceptant } B(R+y) \setminus B(R)\} \\
&\leq N'_{R+y,k-1} + N'_{R,k-1} + \#\{(k-1)\text{-faces incluses dans } B(R+y) \setminus B(R)\} \\
&\leq N'_{R+y,k-1} + N'_{R,k-1} + \binom{d+1}{k-1} \cdot \#\{0\text{-faces incluses dans } B(R+y) \setminus B(R)\} \\
&\leq N'_{R+y,k-1} + N'_{R,k-1} + \binom{d+1}{k-1} (S_{R+y,0} - S_{R,0}).
\end{aligned} \tag{1.19}$$

En utilisant la convergence $(C)_{k-1}$ ainsi que le lemme 1.3.5 appliqué à $k = 0$, nous déduisons que

$$\frac{\#\{(k-1)\text{-faces interceptant } B(R+y)\setminus B(R)\}}{v(R)} \to 0 \text{ p.s. quand } R \to +\infty. \qquad (1.20)$$

En combinant les inégalités (1.16) et (1.18), nous obtenons

$$\frac{N'_{R,k}}{v(R)} \leq \frac{1}{A_k}\frac{S_{R+y,k}-S_{R,k}}{v(R)} + \frac{d-k+2}{v(R)}\#\{(k-1)\text{-faces interceptant } B(R+y)\setminus B(R)\}.$$

Compte tenu du lemme 1.3.5 et de (1.20), cette dernière inégalité implique la convergence presque-sûre $(C)_k$. Le lemme 1.3.6 est ainsi démontré.

□

Remarque 1.3.7 Le lemme 1.3.4 implique en particulier que le reste $\varepsilon(R, \mathbf{1}, \cdot)$ dans (1.10) tend vers zéro p.s., c'est-à-dire

$$\frac{N_R}{v(R)} \to \mathbf{E}\left(\frac{1}{V_d(C_0)}\right) \quad \textit{p.s.} \quad \textit{quand } R \to +\infty.$$

En reprenant mot pour mot la preuve du lemme 4 de [26] dûe à A. Goldman, nous obtenons le corollaire suivant.

Corollaire 1.3.8 *Soit $h : \mathcal{K} \longrightarrow \mathbb{R}$ une application mesurable telle qu'il existe $p > 1$ satisfaisant $\mathbf{E}|h(\mathcal{C})|^p < +\infty$. Alors presque sûrement,*

$$\widetilde{\mathbf{E}}h = \lim_{R\longrightarrow\infty}\frac{1}{N_R}\sum_{C\in\mathcal{C}_R} h(C) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)}\mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right).$$

1.4 La mosaïque de Johnson-Mehl.

1.4.1 La cellule typique au sens de Palm.

La cellule typique $\mathcal{C}$ au sens de Palm de $\mathcal{J}_d$ a été introduite par J. Møller par analogie avec le cas Voronoi. Plus précisément, fixons un ensemble borélien borné $B \subset \mathbb{R}^d$ de mesure de Lebesgue non nulle. Notons $\overline{\Phi}$ l'ensemble des couples (x, t) tels que la cellule associée est non vide. Classiquement, $\mathcal{C}$ est définie par l'identité [64] :

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\lambda V_d(B)}\mathbf{E}\sum_{(x,t)\in\overline{\Phi};x\in B} h(C((x,t))-x),$$

où $h : \mathcal{K} \longrightarrow \mathbb{R}$ parcourt l'ensemble des fonctions numériques, mesurables et bornées.

Considérons par ailleurs un temps aléatoire τ dont la loi a pour densité par rapport à la mesure Λ, $p(t)/\lambda$. Alors en utilisant la formule de Slivnyak [64], la cellule

$$C((0,\tau)) = \{y \in \mathbb{R}^d; T_{(0,\tau)} \leq T(x,t), (x,t) \in \Phi\},$$

obtenue en rajoutant le couple $(0,\tau)$ au processus ponctuel Φ, coïncide en loi avec la cellule typique $\mathcal{C}$.

Par ailleurs, en adaptant mot pour mot le raisonnement de Møller ([65], Prop. 3.3.2.) sur les mosaïque de Poisson-Voronoi, on obtient que la cellule $\mathcal{C}$ est reliée en loi à la cellule contenant l'origine C_0 de la manière suivante :

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}(1/V_d(C_0))}\mathbf{E}\left\{\frac{h(C_0)}{V_d(C_0)}\right\},$$

pour toute fonction mesurable bornée $h : \mathcal{K} \longrightarrow \mathbb{R}$ invariante par translation.

1.4.2 Convergence des moyennes empiriques.

L'ergodicité de $\mathcal{J}_d$ s'obtient de manière analogue à celle de $\mathcal{V}_d$. Par ailleurs, définissons de même que dans la preuve de la proposition 1.3.4 les grandeurs $N_{R,k}$, $N'_{R,k}$ et $S_{R,k}$, $0 \le k \le d$, $R \ge 0$. Le raisonnement complet conduisant à la convergence des moyennes empiriques peut alors être repris, à condition que toutes ces variables soient intégrables. On peut facilement vérifier qu'il suffit de se donner l'hypothèse de l'intégrabilité du coefficient binomial $\binom{N_R+N'_R}{d+1}$. La proposition suivante fournit une condition explicite sur la mesure Λ pour que cette hypothèse soit vérifiée.

Proposition 1.4.1 *Si la condition*

$$\int_0^{+\infty}\left(\int_0^{t+K}(t+K-s)^d d\Lambda(s)\right)^{d+1}\left(\int_0^t (t-s)^{d-1}d\Lambda(ds)\right)p(t)dt < +\infty \tag{1.21}$$

est vérifiée pour tout $K > 0$, alors $\mathbf{E}\binom{N_R+N'_R}{d+1} < +\infty$.

Preuve. Considérons T_0 le premier temps d'atteinte de l'origine par un germe en croissance. La loi de T_0 est donnée par l'égalité [64]

$$\mathbf{P}\{T_0 \ge t\} = p(t), \tag{1.22}$$

où p est la fonction définie par (1.3).

De plus, on peut montrer que conditionnellement à l'évènement $\{T_0 = t\}$, $t > 0$, le processus ponctuel Φ est égal en loi à $\Phi_t \cup \{(X,T)\}$ où :
(i) Φ_t est un processus ponctuel de Poisson sur $\mathbb{R}^d \times \mathbb{R}_+$, de mesure d'intensité

$$\mathbf{1}_{D_t^c}(x,t)dx\Lambda(dt),$$

avec

$$D_t = \{(x,s) \in \mathbb{R}^d \times \mathbb{R}_+; T_{(x,s)}(0) \le t\};$$

(ii) (X,T) est un point indépendant du processus précédent, de loi uniforme sur la frontière du cône D_t.

De plus, si la cellule associée à un germe (x,s) intercepte $B(R)$, alors nécessairement,

$$s + (||x|| - R) \le T_0 + R.$$

Par conséquent,

$$\begin{aligned}
&\mathbf{E}\left\{\binom{N_R+N'_R}{d+1}\right\}\\
&\qquad\le \int_0^{+\infty}\mathbf{E}\left[\binom{(\#\{(x,s)\in\Phi; ||x||+s\le t+2R\})}{d+1}\middle| T_0=t\right]\mathbf{P}\{T_0\in dt\}\\
&\qquad\le \int_0^{+\infty}\mathbf{E}\left[\binom{(\#\{(x,s)\in\Phi; ||x||+s\le t+2R\}+1)}{d+1}\right]\mathbf{P}\{T_0\in dt\}. \quad (1.23)
\end{aligned}$$

Comme la variable $\#\{(x,s)\in\Phi; ||x||+s\le t+2R\}$ est de Poisson de paramètre

$$a_{t,R}=\int \mathbf{1}_{\{||x||+s\le t+2R\}}dx\Lambda(ds)=\omega_d\int_0^{t+2R}(t+2R-s)^d\Lambda(ds),$$

on a

$$\mathbf{E}\left[\binom{(\#\{(x,s)\in\Phi; ||x||+s\le t+2R\}+1)}{d+1}\right]=a_{t,R}^{d+1}+(d+1)a_{t,R}^d. \quad (1.24)$$

De plus, $a_{t,R}$ est croissant en R à t fixé et tend vers $+\infty$ quand $R\to+\infty$ (en utilisant la condition (1.1)). Ainsi pour R suffisamment grand, il suffit d'après le résultat (1.24) que l'intégrale $\int_0^{+\infty}a_{t,R}^{d+1}\mathbf{P}\{T_0\in dt\}$ soit finie pour déduire de l'inégalité (1.23) que $\mathbf{E}\binom{N_R+N'_R}{d+1}<+\infty$.

En remplaçant la loi de T_0 fournie par son expression (1.22), la condition précédente se ramène à

$$\int_0^{+\infty}\left(\int_0^{t+2R}(t+2R-s)^d d\Lambda(s)\right)^{d+1}\left(\int_0^t(t-s)^{d-1}d\Lambda(ds)\right)p(t)dt<+\infty,$$

ce qui est bien vérifié sous l'hyothèse (1.21).

□

1.5 La mosaïque poissonienne.

1.5.1 Convergence des moyennes empiriques.

La convergence presque-sûre des moyennes empiriques en toute dimension est déjà connue (voir [26], [72]). Nous en proposons une autre démonstration qui se base sur les idées développées par R. Cowan dans le cas de la dimension deux [15][16]. La technique que nous allons adopter est analogue à celle employée pour la mosaïque de Poisson-Voronoi $\mathcal{V}_d$.

Rappelons tout d'abord comment montrer l'ergodicité de la mosaïque $\mathcal{P}_d$.

On remarque que pour tout $a\in\mathbb{R}^d$,

$$a+H(x_i)=H(t^a(x_i))\quad\text{avec } t^a(x_i)=\left(1+\frac{x_i\cdot a}{||x_i||^2}\right)x_i, i\ge 1.$$

En réalisant Ω comme l'espace $\mathcal{M}_\sigma$ muni de la tribu cylindrique $\mathcal{T}_c$, la correspondance

$$\{x_i; i \geq 1\} \longmapsto \{t^a(x_i); i \geq 1\}$$

induit classiquement une transformation $T^a : \Omega \longrightarrow \Omega$ préservant la mesure [28].

Lemme 1.5.1 *Pour tout $a \in \mathbb{R}^d$, les transformations T^a sont ergodiques.*

Preuve. Il suffit de montrer que la mesure est fortement mélangeante, c'est-à-dire que pour tous $A, B \in \mathcal{B}(\mathbb{R}^d)$, $k, l \in \mathbb{N}$, nous avons la convergence, lorsque $n \to +\infty$,

$$\begin{aligned}
&\mathbf{P}[\{\#(T^{-na}(\Phi) \cap A) = k\} \cap \{\#(\Phi \cap B) = l\}] \\
&\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\}] \\
&\qquad \to \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\}.
\end{aligned} \tag{1.25}$$

Soit $D_\alpha = \{x \in \mathbb{R}^d; |x \cdot a| \geq \alpha\}$, $\alpha > 0$.
- Supposons qu'il existe $\alpha > 0$ tel que $B \subset D_\alpha$. Alors pour n suffisamment grand, $t^{na}(A) \cap B = \emptyset$. Ainsi, comme Φ est un processus poissonien, nous obtenons

$$\begin{aligned}
&\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\}] \\
&\qquad = \mathbf{P}\{\#(\Phi \cap t^{na}(A)) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\} \\
&\qquad = \mathbf{P}\{\#(T^{-na}(\Phi) \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\} \\
&\qquad = \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\}.
\end{aligned}$$

- Dans le cas général, soit $\varepsilon \in (0,1)$ et prenons $\alpha > 0$ tel que l'évènement $E_\alpha = \{\Phi \cap D_\alpha^c \cap B = \emptyset\}$ satisfait

$$P(E_\alpha) \geq 1 - \varepsilon. \tag{1.26}$$

En appliquant le premier cas à $B \cap D_\alpha$, nous obtenons alors que pour n suffisamment grand,

$$\begin{aligned}
&\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\} \cap E_\alpha] \\
&\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B \cap D_\alpha) = l\} \cap E_\alpha] \\
&\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] \cdot \mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\}.
\end{aligned} \tag{1.27}$$

De plus, en utilisant (1.26), nous avons

$$\begin{aligned}
&|\mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\} - \mathbf{P}\{\#(\Phi \cap B) = l\}| \\
&\qquad \leq |\mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\} - \mathbf{P}[\{\#(\Phi \cap B \cap D_\alpha) = l\} \cap E_\alpha]| \\
&\qquad\qquad + |\mathbf{P}[\{\#(\Phi \cap B) = l\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap B) = l\}| \\
&\qquad \leq 2\varepsilon.
\end{aligned} \tag{1.28}$$

Par ailleurs,

$$\begin{aligned}
&|\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap A) = k\}| \\
&\qquad = |\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap t^{na}(A)) = k\}| \leq \varepsilon.
\end{aligned} \tag{1.29}$$

Par conséquent, pour n suffisamment grand, en combinant (1.27), (1.28) et (1.29), nous obtenons que

$$|\mathbf{P}[\{\#(\Phi\cap t^{na}(A)) = k\}\cap\{\#(\Phi\cap B) = l\}] - \mathbf{P}\{\#(\Phi\cap A) = k\}\cdot\mathbf{P}\{\#(\Phi\cap B) = l\}| \leq 4\varepsilon,$$

ce qui prouve la convergence (1.25).

□

Notons $\mathcal{C}_R$ (resp. $\mathcal{C}'_R$) l'ensemble des cellules incluses dans $B(R)$ (resp. interceptant la frontière de $B(R)$) et $N_R = \#\mathcal{C}_R$ (resp. $N'_R = \#\mathcal{C}'_R$).

Prouvons l'intégrabilité de $(N_R+N'_R)$. L'ensemble des hyperplans de $\mathcal{H}(\Phi)$ interceptant $B(R)$ divise presque sûrement l'espace en exactement $2^{\#(\Phi\cap B(R))}$ composantes connexes. Par conséquent, puisque $\#(\Phi\cap B(R))$ est une variable de Poisson,

$$\mathbf{E}(N_R + N'_R) \leq \mathbf{E}2^{\#(\Phi\cap B(R))} < +\infty.$$

Considérons $h : \mathcal{K} \longrightarrow \mathbb{R}_+$ une fonction mesurable, invariante par translation et bornée. De même que pour $\mathcal{V}_d$, les égalités $C_0(T^{-a}\omega) = C_a(\omega) - a$, $a \in \mathbb{R}^d$, $\omega \in \Omega$, permettent d'obtenir les identités presque-sûres (1.10) et (1.11).

En supposant la contribution des restes négligeable, une application du théorème ergodique de Wiener nous fournit le théorème suivant.

Théorème 1.5.2 *Pour toute application h mesurable, positive, bornée et invariante par translation, presque sûrement,*

$$\widetilde{\mathbf{E}}h = \lim_{R\longrightarrow\infty}\frac{1}{N_R}\sum_{C\in\mathcal{C}_R} h(C) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)}\mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right). \tag{1.30}$$

De même que dans la partie 1.3.2, il suffit pour prouver le théor‘eme 1.5.2 d'établir la proposition suivante.

Proposition 1.5.3 *Quand $R \to +\infty$, nous avons :*

$$N'_R/v(R) \to 0 \quad p.s..$$

Preuve. Nous reprenons les mêmes notations $N_{R,k}$, $N'_{R,k}$ et $S_{R,k}$, $R > 0$, $0 \leq k \leq d$, que dans la preuve du lemme 1.3.4.

Prouvons que toutes ces variables sont intégrables. L'ensemble des hyperplans de $\mathcal{H}(\Phi)$ interceptant $B(R)$ induit au plus $\left(2^{\#(\Phi\cap B(R))}\binom{\#(\Phi\cap B(R))}{d-k}\right)$ k-faces de $\mathcal{P}_d$, $0 \leq k \leq d$. Par conséquent, puisque $\#(\Phi\cap B(R))$ est une variable de Poisson,

$$\mathbf{E}(N_{R,k} + N'_{R,k}) \leq \mathbf{E}\left(2^{\#(\Phi\cap B(R))}\binom{\#(\Phi\cap B(R))}{d-k}\right) < +\infty.$$

Par ailleurs, nous avons clairement l'inégalité

$$S_{R,k} \leq \frac{\sigma_k}{k}R^k(N_{R,k} + N'_{R,k}) \text{ p.s.},$$

donc $S_{R,k}$, $0 \leq k \leq d$, est également intégrable.

On déduit la proposition 1.5.3 de deux lemmes intermédiaires :

Lemme 1.5.4 *Pour tout $0 \leq k \leq d$, $S_{R,k}/v(R)$ converge p.s. vers une constante lorsque R tend vers l'infini.*

Lemme 1.5.5 *Pour tout $0 \leq k \leq d$, $N'_{R,k}/v(R)$ converge vers 0 p.s. lorsque R tend vers l'infini.*

La démonstration du lemme 1.5.4 est en tout point analogue à celle du lemme 1.3.5.

Quant à la preuve du lemme 1.5.5, elle est identique à celle du lemme 1.3.6, à ceci près que deux estimées diffèrent : presque sûrement, on a

$$\begin{aligned}
&\#\{(k-1)\text{-faces interceptant } B(R+y) \setminus B(R)\} \\
&\qquad\qquad \leq N'_{R+y,k-1} + N'_{R,k-1} + 2^{k-1}\binom{d}{k-1}(S_{R+y,0} - S_{R,0}),
\end{aligned}$$

et

$$\begin{aligned}
\frac{N'_{R,k}}{v(R)} &\leq \frac{1}{A_k}\frac{S_{R+y,k} - S_{R,k}}{v(R)} \\
&\qquad + \frac{2(d-k+1)}{v(R)}\#\{(k-1)\text{-faces interceptant } B(R+y) \setminus B(R)\}.
\end{aligned}$$

Remarque 1.5.6 Le lemme 1.5.3 implique en particulier que le reste $\varepsilon(R, \mathbf{1}, \cdot)$ dans (1.10) tend vers zéro p.s., c'est-à-dire

$$\frac{N_R}{v(R)} \to \mathbf{E}\left(\frac{1}{V_d(C_0)}\right) \quad \textit{p.s.} \quad \textit{quand } R \to +\infty.$$

A. Goldman ([26], lemme 4) a prouvé un corollaire important en dimension deux qui se généralise aisément.

Corollaire 1.5.7 (Goldman, 1996) *Soit $h : \mathcal{K} \longrightarrow \mathbb{R}$ une application mesurable telle qu'il existe $p > 1$ satisfaisant $\mathbf{E}(|h(C_0)|^p / V_d(C_0)) < +\infty$. Alors presque sûrement,*

$$\widetilde{\mathbf{E}}h = \lim_{R \longrightarrow \infty} \frac{1}{N_R} \sum_{C \in \mathcal{C}_R} h(C) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right).$$

1.5.2 La cellule typique au sens de Palm.

L'objet obtenu par convergence des moyennes empiriques, communément appelé cellule empirique, est relativement peu maniable. Remarquons que dans le cas de la mosaïque de Poisson-Voronoi $\mathcal{V}_d$, une approche en terme de mesure de Palm a permis à l'aide de la formule de Slivnyak de disposer d'une réalisation explicite de la cellule typique.

Aussi, par analogie, nous allons donner une nouvelle construction de cellule typique au sens de Palm pour la mosaïque poissonienne $\mathcal{P}_d$ qui sera équivalente en loi à la cellule empirique.

Pour ce faire, désignons par Ψ le processus ponctuel constitué des centres des disques inscrits dans les cellules de la mosaïque. L'invariance de la mosaïque $\mathcal{P}_d$ par translation implique que Ψ est un processus ponctuel (localement fini) stationnaire. Fixons un borélien $B \subset \mathbb{R}^d$ vérifiant $0 < V_d(B) < +\infty$. La cellule typique $\mathcal{C}$, prise au sens de Palm, est définie par la formule :

$$\mathbf{E}h(\mathcal{C}) = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \frac{1}{V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(C(z) - z), \tag{1.31}$$

où $h : \mathcal{K} \longrightarrow \mathbb{R}$ parcourt l'ensemble des fonctions mesurables bornées (la constante $\omega_d {\omega_{d-1}}^d / 2^d$ est l'intensité du processus Ψ).

En reprenant le raisonnement de J. Møller s'appliquant aux processus ponctuels stationnaires (voir [65], page 66), on obtient dans le cas où h est également invariant par translation, l'égalité

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right).$$

Avec le théorème 1.5.2, on en déduit que :

Théorème 1.5.8 *Pour toute fonction h mesurable, invariante par translation et bornée, on a*

$$\widetilde{\mathbf{E}}h = \mathbf{E}h(\mathcal{C}).$$

L'identité en loi décrite par le théorème précédent (associée à une utilisation du théorème de Slivnyak) aura de multiples conséquences sur nos connaissances des propriétés géométriques de la cellule empirique de la mosaïque poissonienne $\mathcal{P}_d$ (voir le chapitre 4).

Chapitre 2

Résultats sur les lois des caractéristiques géométriques des mosaïques de Poisson-Voronoi et poissoniennes de droites dans le plan.

2.1 Introduction et présentation des principaux résultats.

2.1.1 Notations et contexte.

Dans cette partie, nous nous limiterons au cas de la dimension $d = 2$. On notera $\mathcal{C}$ la cellule typique de Poisson-Voronoi et $C(0)$ la cellule associée à un germe placé en l'origine lorsque l'on rajoute ce germe au processus de départ. Par ailleurs, C'_0 (resp. $\mathcal{C}'$) désignera la cellule de Crofton (resp. la cellule typique) d'une mosaïque poissonienne de droites. Enfin, $D(y, r)$ sera le disque centré en un point $y \in \mathbb{R}^2$, de rayon $r \geq 0$, $V_1(\cdot)$ le périmètre d'un ensemble convexe compact et $N_0(\cdot)$ le nombre de sommets d'un polygone convexe.

Rappelons que la cellule typique de Poisson-Voronoi est équivalente en loi à la cellule $C(0)$. Ainsi pour la décrire, il suffit de connaître les positions relatives des médiatrices des segments $[0, x]$, $x \in \Phi$, bordant $C(0)$. De même, pour connaître la géométrie de la cellule de Crofton C'_0 d'une mosaïque poissonienne, nous devons étudier les positions relatives des droites $H(x)$, $x \in \Psi$, bordant C'_0. Ainsi, on peut raisonnablement penser que des techniques analogues peuvent s'appliquer dans les deux cas.

2.1.2 Résultats connus sur les lois des caractéristiques géométriques fondamentales de la cellule typique de Poisson-Voronoi.

Jusqu'à présent, peu de lois explicites de caractéristiques de la cellule $\mathcal{C}$ ont été obtenues, ce qui explique le recours intensif à des techniques de simulation (voir [69] pour un état des lieux à peu près exhaustif de l'ensemble des approximations déduites de simulation). Cela étant, il est facile de déterminer la loi du rayon R_m du plus grand disque centré en l'origine contenu dans $C(0)$:

$$\mathbf{P}\{R_m \geq r\} = e^{-4\pi r^2}, \quad r \geq 0.$$

La question de la loi du rayon R_M du plus petit disque centré à l'origine contenant $C(0)$ est plus délicate. S. G. Foss et S. A. Zuyev ont obtenu un majorant de la queue de R_M dans le cadre de l'étude d'un réseau de télécommunications [21] :

$$\mathbf{P}\{R_M \geq r\} \leq 7e^{-\mu r^2}, \quad r > 0,$$

où $mu = 2(\sin(\pi/14)\cos(5\pi/14) + \pi/7) \approx 1.09$.

En 1961, E. N. Gilbert [23] a fourni le meilleur encadrement à ce jour de la queue de l'aire de $\mathcal{C}$:

$$e^{-4t} \leq \mathbf{P}\{V_2(\mathcal{C}) \geq t\} \leq \frac{t-1}{e^{t-1}-1}, \quad t > 0. \tag{2.1}$$

Plus récemment, S. A. Zuyev [94] a prouvé en s'appuyant sur la formule de Russo que conditionnellement à l'évènement $\{N_0(C(0)) = k\}$, $k \geq 3$, l'aire du domaine fondamental de $C(0)$, c'est-à-dire de l'ensemble $\cup_{x\in C(0)} D(x, ||x||)$, suit une loi Gamma de paramètres $(k, 1)$. Enfin, A. Hayen et M. Quine [33] ont fourni une expression intégrale explicite pour la probabilité $\mathbf{P}\{N_0(\mathcal{C}) = 3\}$.

Enfin, les deux premiers moments de l'aire, du périmètre et du nombre de sommets ont été calculés [65].

2.1.3 Résultats connus sur les lois des caractéristiques géométriques fondamentales des cellules typique et de Crofton d'une mosaïque poissonienne de droites.

R. E. Miles a exhibé dès 1963 [54] [55], au moyen de raisonnements en partie heuristiques, les principaux résultats en loi connus actuellement : les rayons du disque inscrit de la cellule typique $\mathcal{C}'$ et du plus petit disque centrés à l'origine inclus dans C'_0 sont identiques en loi et suivent la loi exponentielle de paramètre 2π. Conditionnellement à l'évènement $\{N_0(\mathcal{C}') = k\}$, $k \geq 3$, $V_1(\mathcal{C}')$ suit une loi Gamma de paramètres $(k-2, 1)$. Plus récemment, l'estimation asymptotique des queues des aires de $\mathcal{C}'$ et C'_0 a été étudiée par A. Goldman [28], puis par I. N. Kovalenko [41], [42]. En particulier,

$$\lim_{t\to+\infty} t^{-1/2} \ln \mathbf{P}\{V_2(\mathcal{C}') \geq t\} = \lim_{t\to+\infty} t^{-1/2} \ln \mathbf{P}\{V_2(C'_0) \geq t\} = -2\sqrt{\pi}. \tag{2.2}$$

Par ailleurs, les probabilités $\mathbf{P}\{N_0(\mathcal{C}) = 3\} = 2 - \pi^2/6$, et $\mathbf{P}\{N_0(\mathcal{C}) = 4\}$ ont respectivement été déterminées par Miles [54] et Tanner [89].

Ajoutons que les deux premiers moments de $V_2(\mathcal{C}')$ ont été calculés en 1945 par S. A. Goudsmit [31], puis ceux de $V_1(\mathcal{C}')$ et $N_0(\mathcal{C}')$ par Miles en 1964 [54]. Depuis, J. C. Tanner a complété cette étude [89] en fournissant des moments d'ordre trois et quatre. Concernant la cellule de Crofton C'_0, G. Matheron [48] a déterminé les moyennes de $V_1(C'_0)$ et $N_0(C'_0)$.

Enfin, signalons que des théorèmes centraux-limites ont été obtenus dans ce contexte par K. Paroux [72].

2.1.4 Présentation des nouveaux résultats.

Formules intégrales pour les lois des principales caractéristiques géométriques.

Miles et Maillardet [60] ont donné par un argument heuristique une description "peu maniable" de la loi de $N_0(\mathcal{C})$ ne permettant pas de déboucher sur des expressions analytiques précises. En 2000, Hayen et Quine ont obtenu par une méthode différente une expression intégrale de $\mathbf{P}\{N_0(\mathcal{C}) = 3\}$.

Dans un premier article, nous obtenons une formule intégrale analytique de la loi de $N_0(\mathcal{C})$. Pour cela, nous faisons appel à la formule de Slivnyak ainsi qu'à des arguments géométriques décrivant la forme du domaine fondamental d'un polygone.

Ces formules permettent d'accéder, via l'implémentation appropriée des calculs intégraux sur ordinateur, aux valeurs numériques approchées des probabilités $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 3$, avec la précision souhaitée. Jusqu'à présent, ces valeurs étaient obtenues par la technique de simulation aléatoire de la mosaïque, ce qui ne fournissait en aucun cas le moyen d'estimer la marge d'erreur.

Dans un second article, une méthode analogue nous permet d'obtenir les lois explicites du vecteur formé par les positions (angle d'inclinaison, distance à l'origine) des côtés de $C(0)$ (resp. de C_0' et $\mathcal{C}'$), de l'aire et du périmètre de $\mathcal{C}$ (resp. de C_0' et $\mathcal{C}'$), de l'aire du domaine fondamental de $C(0)$ et du nombre de côtés de C_0' et $\mathcal{C}'$. Plus précisément, nous déterminons conditionnellement à $\{N_0(C(0) = k\}$, $k \geq 3$: les lois du vecteur des positions des côtés, de $V_2(C(0))$, $V_1(C(0))$ et $V_2(\cup_{x\in C(0)} D(x, ||x||))$. En particulier, nous retrouvons le résultat de Zuyev selon lequel conditionnellement au nombre de côtés, l'aire du domaine fondamental suit une loi Gamma.

La même méthode s'applique à la cellule de Crofton C_0'. Le rôle joué par l'aire du domaine fondamental intervenant dans la formule intégrale de la loi du nombre de côtés de la cellule typique de Poisson-Voronoi est à présent tenu par le périmètre d'un polygone à k côtés, $k \geq 3$. Son calcul en fonction des coordonnées polaires des côtés pose nettement moins de difficultés. Nous obtenons ainsi une expression explicite de la loi du nombre de côtés $N_0(C_0')$, et sachant $\{N_0(C_0') = k\}$, $k \geq 3$: les lois des positions des côtés bordant C_0', de l'aire $V_2(C_0')$ et du périmètre $V_1(C_0')$. En particulier, $V_1(C_0')$ est de loi Gamma de paramètres $(k, 1)$.

Enfin, comme on dispose de la distribution conjointe du couple $(N_0(C_0), V_2(C_0))$, on peut utiliser l'égalité (1.30) reliant la loi de la cellule typique à celle de la cellule de Crofton C_0' pour obtenir dans le cas de $\mathcal{C}'$, la loi du nombre de côtés et les lois conditionnelles déjà citées. En particulier, conditionnellement à $\{N_0(\mathcal{C}') = k\}$, $k \geq 3$, on retrouve le fait dû à Miles selon lequel $V_1(\mathcal{C}')$ suit une loi Gamma de paramètres $(k-2, 1)$.

La loi de la plus petite couronne centrée en l'origine contenant le bord de $C(0)$ (resp. C_0').

Afin d'étudier la forme de la cellule typique réalisée en $C(0)$, on définit le rayon R_m (resp. R_M) du plus grand (resp. petit) disque centré à l'origine inclus dans (resp. contenant) $C(0)$. La frontière de $C(0)$ est alors contenue dans la couronne (aléatoire) $D(0, R_M) \setminus D(0, R_m)$.

Nous donnons la loi conjointe du couple (R_m, R_M) en reliant ce problème à une question de recouvrement du cercle par des arcs aléatoires indépendants et identiquement distribués, dont les centres sont uniformément répartis et les longueurs suivent une loi fixée ν. Ce domaine des probabilités a été largement développé, notamment par W. L. Stevens [85], A. F. Siegel [81], L. A. Shepp [79] et J. P. Kahane [39]. La loi conditionnelle de R_M sachant $\{R_m = r\}$, $r \geq 0$, s'exprime en termes de probabilités de recouvrement du cercle $P(\nu, n)$ par n arcs i.i.d., $n \geq 1$, dont les longueurs ont pour loi ν. Le calcul explicite de $P(\nu, n)$ est fourni par le travail de A. F. Siegel et L. Holst [82]. Néanmoins, la formule est peu maniable. De ce fait, nous déterminons des encadrements de la queue de la loi de R_M. Pour cela, nous sommes conduits à comparer des probabilités de recouvrement des probabilités de recouvrement $P(\nu_1, n)$ et $P(\nu_2, n)$, où ν_1 et ν_2 sont des lois distinctes de même espérance. Cela nous a amené à résoudre une conjecture de Siegel. En particulier, nous obtenons

$$2\pi r^2 e^{-\pi r^2} \leq \mathbf{P}\{R_M \geq r\} \leq 4\pi r^2 e^{-\pi r^2}, \quad r \geq \alpha \approx 0.337,$$

et pour tous $0 < c < 8/(3\sqrt{2})$, $-1 < \alpha < 1/3$,

$$\mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} = O(e^{-cr^{\frac{1}{2}(1-3\alpha)}}), \quad \text{quand } r \to +\infty.$$

Ce dernier résultat implique que conditionnellement à $\{R_m = r\}$, la frontière de la cellule typique se trouve incluse avec une “forte probabilité”, lorsque r est “grand”, dans une couronne d'épaisseur de l'ordre de $r^{-1/3}$. Autrement dit, les grandes cellules de la mosaïque de Poisson-Voronoi, au sens où elles ont un grand rayon de disque inscrit, ont une forme approximativement circulaire. Ce fait peut être observé lors des simulations.

Le même raisonnement est valable pour l'étude du couple (R'_m, R'_M), où R'_m (resp. R'_M) est le rayon du plus grand (resp. petit) disque centré à l'origine contenu dans (resp. contenant) la cellule de Crofton C'_0.

The explicit expression of the distribution of the number of sides of the typical Poisson-Voronoi cell. *

Pierre Calka†

Abstract

In this paper, we give an explicit expression for the distribution of the number of sides (or equivalently vertices) of the typical cell of a two-dimensional Poisson-Voronoi tessellation. We use this formula to give a table of numerical values of the distribution function.

1 Introduction and principal result.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^2$, with the two-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^2; ||y - x|| \leq ||y - x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [8] and Gilbert [3] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [7], telecommunications [1], astrophysics [14] and ecology [12]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [13] and Okabe et al. [11].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [10]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [10]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B \cap \Phi} h(C(x) - x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^2; ||y|| \leq ||y - x||, x \in \Phi\}$$

**American Mathematical Society 2000 subject classifications.* Primary 60D05; secondary 60G55.
Key words and phrases. Palm distribution, Poisson-Voronoi tessellation, stochastic geometry, typical cell.

†*Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr

obtained when the origin is added to the point process Φ. It is well known [10] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$.

The explicit distributions of the main geometrical characteristics of the typical cell, as the area or the perimeter, are mainly unknown. Nevertheless, the law of the largest disk centered at the origin and contained in $C(0)$ is easy to obtain and the distribution of the radius of the smallest disk centered at the origin and containing $C(0)$ has been recently calculated [2].

Let $N_0(\mathcal{C})$ denote the number of sides (or equivalently vertices) of the typical cell $\mathcal{C}$. Miles and Maillardet [9] obtained integral formulas for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 3$, but did not precisely evaluate them. These probabilities have been often estimated by simulation (see for example [11], Table 5.5.13). Moreover, an explicit integral formula has been recently obtained by Hayen and Quine [6] for $\mathbf{P}\{N_0(\mathcal{C}) = 3\}$. However, it seems to be difficult to generalize their method for getting $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 4$.

In this paper, we use a technique based on the famous formula due to Slivnyak (see for example [10]) to provide an explicit expression of the distribution of $N_0(\mathcal{C})$.

Theorem 1 *For every $k \geq 3$, we have*

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) p_i dp_i, \tag{1}$$

where σ_k is the (normalized) uniform measure on the simplex

$$\{(\delta_1, \cdots, \delta_k) \in [0, 2\pi]; \sum_{i=1}^{k} \delta_i = 2\pi\}, \tag{2}$$

with

$$B = \{(p, q, r, \alpha, \beta) \in (\mathbb{R}_+)^3 \times (0, \pi)^2; p \sin(\beta) + r \sin(\alpha) \geq q \sin(\alpha + \beta)\}, \tag{3}$$

with for every $\delta \in (0, \pi)$, $p, q \geq 0$,

$$H(\delta, p, q) = \frac{1}{2\sin^2(\delta)} \left\{ (p^2 + q^2 - 2pq\cos(\delta))\frac{\delta}{2} + pq\sin(\delta) - \frac{p^2}{4}\sin(2\delta) - \frac{q^2}{4}\sin(2\delta) \right\}, \tag{4}$$

and with the conventions $p_0 = p_k$, $p_{k+1} = p_1$, and $\delta_0 = \delta_k$.

Remark 1. The method used in the proof of Theorem 1 also gives the joint distribution of the respective positions of the k lines bounding the typical cell $\mathcal{C}$ conditionally to the event $\{N_0(\mathcal{C}) = k\}$, $k \geq 3$. This provides in particular a new proof of the following result due to Zuyev [15]: conditionally to $\{N_0(\mathcal{C}) = k\}$, the area of the fundamental domain of $\mathcal{C}$ is Gamma distributed, of parameters (k,1). Besides, we can adapt the procedure in order to get the distribution of the number of sides of the Crofton cell of an isotropic Poisson line process in the plane. All these results will take place in a future paper.

Remark 2. We can rewrite $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $k \geq 3$, as an expectation . Let us

k	3	4	5	6	7	8	9	10
$\mathbf{P}\{N_0(\mathcal{C}) = k\}$	0.011	0.107	0.260	0.294	0.197	0.092	0.028	0.006

Table 1: Numerical values for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$.

consider a random vector $(\Delta_1, \cdots, \Delta_k)$ uniformly distributed on the simplex given in (2), and a vector $(Q_1, \cdots, Q_k)$ such that conditionally to $\{\Delta_1 = \delta_1, \cdots, \Delta_k = \delta_k\}$, $Q_1, \cdots, Q_k$ are independent and exponentially distributed, with respective parameters $f(\delta_1, \delta_k)$, $f(\delta_2, \delta_1), \cdots, f(\delta_k, \delta_{k-1})$ where

$$f(x,y) = \frac{1}{4\sin^2(x)}\left(x - \frac{\sin(2x)}{2}\right) + \frac{1}{4\sin^2(y)}\left(y - \frac{\sin(2y)}{2}\right), \quad x, y \in [0, 2\pi].$$

We then have that for every $k \geq 3$,

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{(2\pi)^k}{k!}\mathbf{E}\left\{\prod_{i=1}^{k} \frac{\exp\left(\frac{\cos(\Delta_i)\Delta_i - \sin(\Delta_i)}{2\sin^2(\Delta_i)}\sqrt{Q_i Q_{i+1}}\right)}{2f(\Delta_i, \Delta_{i-1})}\mathbf{1}_{E_i}\right\}, \tag{5}$$

where the event E_i, $i \geq 1$, is defined by

$$E_i = \{(\sqrt{Q_{i-1}}, \sqrt{Q_i}, \sqrt{Q_{i+1}}, \Delta_{i-1}, \Delta_i) \in B\},$$

and with the conventions $Q_0 = Q_k$, $Q_{k+1} = Q_1$, and $\Delta_0 = \Delta_k$.

Numerical values for $\mathbf{P}\{N_0(\mathcal{C}) = k\}$, $3 \leq k \leq 10$, obtained by a Monte-Carlo procedure using (5) are listed in Table 1. We notice that they are close to the results obtained by simulation given in [11].

The rest of the paper will be devoted to the proof of Theorem 1.

2 Proof of Theorem 1.

For every $x \in \mathbb{R}^2$, let us denote by $L(x)$ (respectively $\mathcal{D}(x)$) the bisecting line of the segment $[0, x]$ (respectively the half-plane containing 0 delimited by $L(x)$).

We then define for all $k \geq 3$, and $x_1, \cdots, x_k \in \mathbb{R}^2$, the domain

$$\mathcal{D}(x_1, \cdots, x_k) = \cap_{i=1}^k \mathcal{D}(x_k).$$

Besides, we consider the set of $(\mathbb{R}^2)^k$

$$A_k = \{(x_1, \cdots, x_k) \in (\mathbb{R}^2)^k; \mathcal{D}(x_1, \cdots, x_k) \text{ is a convex polygon with } k \text{ sides}\}, \tag{6}$$

and for every $(x_1, \cdots, x_k) \in A_k$, the Lebesgue measure

$$V(x_1, \cdots, x_k) = V_2\left[\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)\right], \tag{7}$$

$D(y, r)$ being the disk centered at $y \in \mathbb{R}^2$ and of radius $r > 0$.

Proposition 1 *For every $k \geq 3$,*

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{1}{k!}\int \exp\{-V(x_1,\cdots,x_k)\}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k. \qquad (8)$$

Proof. Using the equality in law $\mathcal{C} \overset{law}{=} C(0)$,

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C}) = k\} &= \mathbf{E}\left\{\sum_{\{x_1,\cdots,x_k\}\subset\Phi} \mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{1}_{\{\mathcal{D}(x_1,\cdots,x_k)=C(0)\}}\right\} \\ &= \mathbf{E}\left\{\sum_{\{x_1,\cdots,x_k\}\subset\Phi} \mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{1}_{\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\setminus\{x_1,\cdots,x_k\}\}}\right\}.\end{aligned}$$

Using Slivnyak's formula [10], we obtain

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C}) = k\} &= \frac{1}{k!}\int \mathbf{1}_{A_k}(x_1,\cdots,x_k)\mathbf{E}\left(\mathbf{1}_{\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}}\right)dx_1\cdots dx_k \\ &= \frac{1}{k!}\int \mathbf{1}_{A_k}(x_1,\cdots,x_k) \\ &\qquad \mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\}dx_1\cdots dx_k. \qquad (9)\end{aligned}$$

We can easily verify that for any $z \in \mathbb{R}^2$,

$$L(z)\cap\mathcal{D}(x_1,\cdots,x_k) \neq \emptyset \Longleftrightarrow z \in \cup_{x\in\mathcal{D}(x_1\cdots,x_k)}D(x,||x||),$$

From this remark and the Poissonian property of Φ, we get

$$\begin{aligned}\mathbf{P}\{L(y)\cap\mathcal{D}(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi\} &= \mathbf{P}\{\Phi\cap\left[\cup_{x\in\mathcal{D}(x_1,\cdots,x_k)}D(x,||x||)\right]=\emptyset\} \\ &= e^{-V(x_1,\cdots,x_k)}. \qquad (10)\end{aligned}$$

Inserting the equality (10) in (9), we deduce Proposition 1.

□

Our next task is to understand analytically the set A_k, $k \geq 3$, and then calculate the area $V(x_1,\cdots,x_k)$ for $(x_1,\cdots,x_k) \in A_k$. To this end, let us denote by (p_i,θ_i), $p_i \geq 0$, $\theta_i \in [0,2\pi)$, the polar coordinates of x_i, $1 \leq i \leq k$. Supposing $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$, we define $\delta_i = \theta_{i+1} - \theta_i$ with the convention $\theta_{k+1} = \theta_1 + 2\pi$. In particular, $\sum_{i=1}^k \delta_i = 2\pi$.

The following lemma gives a necessary and sufficient condition for a vector to be in the set A_k.

Lemma 1 *For every vector $(x_1,\cdots,x_k) \in (\mathbb{R}^2)^k$ such that $0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$,*

$$(x_1,\cdots,x_k) \in A_k \Longleftrightarrow \begin{cases} p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) \geq p_i\sin(\delta_i+\delta_{i-1}) \\ 0 \leq \delta_i \leq \pi. \end{cases} \quad \forall 1 \leq i \leq k,$$

Proof. Let us first notice that having all the angles δ_i, $1 \le i \le k$, less than π is clearly a necessary condition to obtain a bounded convex set. From now on, we suppose this is verified.

$\mathcal{D}(x_1, \cdots, x_k)$ is then a convex polygon with k sides if and only if for every $1 \le i \le k$, the condition (C_i) is satisfied, i.e. the intersection point of $L(x_{i-1})$ and $L(x_{i+1})$ is not between $L(x_i)$ and the parallel line $L_0(x_i)$ containing the origin (with the conventions $x_0 = x_k$ and $x_{k+1} = x_1$).

Because of the invariance by rotation, one may suppose that $L(x_i)$ has the equation $(x = p_i/2)$ and $L_0(x_i)$ is the y-axis. In that case, the equations of the lines $L(x_{i+1})$ and $L(x_{i-1})$ are respectively $(\cos(\delta_i)x + \sin(\delta_i)y - p_{i+1}/2 = 0)$ and $(\cos(\delta_{i-1})x - \sin(\delta_{i-1})y - p_{i-1}/2 = 0)$. We then obtain that the first coordinate of the intersection of these two lines is

$$X_{i-1,i+1} = (p_{i+1}\sin(\delta_{i-1}) + p_{i-1}\sin(\delta_i))/(2\sin(\delta_i + \delta_{i-1})).$$

Consequently, (C_i) holds if and only if

$$\frac{p_{i+1}\sin(\delta_{i-1}) + p_{i-1}\sin(\delta_i)}{2\sin(\delta_i + \delta_{i-1})} \in (-\infty, 0] \cup [\frac{p_i}{2}, +\infty),$$

or equivalently,

$$p_{i+1}\sin(\delta_{i-1}) + p_{i-1}\sin(\delta_i) \ge p_i \sin(\delta_i + \delta_{i-1}).$$

This proves Lemma 1.

□

It remains to determine the area $V(x_1, \cdots, x_k)$, $(x_1, \cdots, x_k) \in A_k$, defined in (7). This is the goal of the next lemma.

Lemma 2 *For every $(x_1, \cdots, x_k) \in A_k$, such that $0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_n$,*

$$\begin{aligned} V(x_1, \cdots, x_k) &= \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \left\{ (p_i^2 + p_{i+1}^2 - 2p_i p_{i+1}\cos(\delta_i))\frac{\delta_i}{2} + p_i p_{i+1}\sin(\delta_i) \right. \\ &\qquad \left. - \frac{p_i^2}{4}\sin(2\delta_i) - \frac{p_{i+1}^2}{4}\sin(2\delta_i) \right\}. \end{aligned}$$

Proof. Supposing that $\mathcal{D}(x_1, \cdots, x_k)$ is a bounded convex set with k sides, let us define by L_θ (respectively l_θ) the support line of $\mathcal{D}(x_1, \cdots, x_k)$ orthogonal to the vector $u_\theta = (\cos(\theta), \sin(\theta))$ (respectively the distance between the origin and L_θ).

We then notice the following equivalence (see [5] and [4] for more details):

$$z \in \left[\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)\right] \cap (\mathbb{R}_+ \cdot u_\theta) \Longleftrightarrow ||z|| \le 2l_\theta.$$

Integrating in polar coordinates, we get that

$$\begin{aligned} V(x_1, \cdots, x_k) &= \int_0^{2\pi} \left(\int_0^{2l_\theta} r dr \right) d\theta \\ &= 2\int_0^{2\pi} {l_\theta}^2 d\theta. \qquad (11) \end{aligned}$$

It remains to determine the distance l_θ. Denoting by $s_1, \cdots, s_k$ the consecutive vertices of the polygon $\mathcal{D}(x_1, \cdots, x_k)$, i.e. $s_i = L(x_{i+1}) \cap L(x_i)$, we have that

$$l_\theta = s_i \cdot u_\theta \quad \forall \theta \in (\theta_i, \theta_{i+1}), \tag{12}$$

where $\cdot$ denotes the usual scalar product of $\mathbb{R}^2$. Besides, by elementar geometric arguments,

$$s_i \cdot \frac{x_i}{||x_i||} = \frac{p_i}{2} \text{ and } s_i \cdot \frac{x_{i+1}}{||x_{i+1}||} = \frac{p_{i+1}}{2}. \tag{13}$$

Inserting the equalities (13) in (12), we obtain that for every $\theta \in (\theta_i, \theta_{i+1})$,

$$l_\theta = \frac{p_i}{2} \cos(\theta - \theta_i) + \left(\frac{1}{\sin(\delta_i)} \frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)} \frac{p_i}{2} \right) \sin(\theta - \theta_i) \tag{14}$$

Consequently, we combine the equality (11) with (14) to deduce

$$\begin{aligned}
V(x_1, \cdots, x_k) &= 2 \sum_{i=1}^{k} \int_0^{\delta_i} l_{\theta+\theta_i}^2 d\theta \\
&= 2 \sum_{i=1}^{k} \int_0^{\delta_i} \left\{ \frac{p_i^2}{4} \cos^2(\theta) + \left(\frac{1}{\sin(\delta_i)} \frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)} \frac{p_i}{2} \right)^2 \sin^2(\theta) \right. \\
&\qquad \left. + \frac{p_i}{2} \left(\frac{1}{\sin(\delta_i)} \frac{p_{i+1}}{2} - \frac{\cos(\delta_i)}{\sin(\delta_i)} \frac{p_i}{2} \right) \sin(2\theta) \right\} d\theta \\
&= \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \left\{ (p_i^2 + p_{i+1}^2 - 2p_i p_{i+1} \cos(\delta_i)) \frac{\delta_i}{2} + p_i p_{i+1} \sin(\delta_i) \right. \\
&\qquad \left. - \frac{p_i^2}{4} \sin(2\delta_i) - \frac{p_{i+1}^2}{4} \sin(2\delta_i) \right\},
\end{aligned}$$

which is the required result of Lemma 2.

□

Proof of Theorem 1. Using polar coordinates in the integral of the equality (8) of Proposition 1, we obtain for every $k \geq 3$,

$$\begin{aligned}
\mathbf{P}\{N_0(\mathcal{C}) = k\} &= \frac{1}{k!} \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} \mathbf{1}_{A_k}(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \\
&\qquad \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} \mathbf{1}_{\{0 \leq \theta_i \leq 2\pi\}} p_i dp_i d\theta_i \\
&= \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} \mathbf{1}_{A_k}(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \\
&\qquad \mathbf{1}_{\{0 \leq \theta_1 \leq \cdots \leq \theta_k \leq 2\pi\}} \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} p_i dp_i d\theta_i
\end{aligned} \tag{15}$$

Inserting then the results of Lemmas 1 and 2 in (15), we deduce that

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C})=k\} &= \int\left[\int\prod_{i=1}^{k}e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i)p_i dp_i\right] \\ &\qquad\qquad \mathbf{1}_{\{\delta_1+\cdots+\delta_{k-1}\le 2\pi\}}\delta_k d\delta_1\cdots d\delta_{k-1} \\ &= \frac{(2\pi)^k}{k!}\int d\sigma_k(\delta_1,\cdots,\delta_k) \\ &\qquad \int\prod_{i=1}^{k}e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i)p_i dp_i,\end{aligned}$$

where the function H (resp. the set B) is defined by the equality (4) (resp. (3)).

This completes the proof of Theorem 1.

□

References

[1] F. Baccelli and B. Błaszczyszyn. On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Adv. in Appl. Probab.*, 33(2):293–323, 2001.

[2] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *To appear in Adv. in Appl. Probab.*, 2002.

[3] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33:958–972, 1962.

[4] A. Goldman and P. Calka. On the spectral function of the Johnson-Mehl and Voronoi tessellations. *Preprint 00-02 of LaPCS*, 2000.

[5] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[6] A. Hayen and M. Quine. The proportion of triangles in a Poisson-Voronoi tessellation of the plane. *Adv. in Appl. Probab.*, 32(1):67–74, 2000.

[7] S. Kumar and R. N. Singh. Thermal conductivity of polycristalline materials. *J. of the Amer. Cer. Soc.*, 78(3):728–736, 1995.

[8] J. L. Meijering. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.*, 8, 1953.

[9] R. E. Miles and R. J. Maillardet. The basic structures of Voronoĭ and generalized Voronoĭ polygons. *J. Appl. Probab.*, (Special Vol. 19A):97–111, 1982. Essays in statistical science.

[10] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[11] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams.* John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[12] E. Pielou. *Mathematical ecology.* Wiley-Interscience, New-York, 1977.

[13] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications.* John Wiley & Sons Ltd., Chichester, 1987. With a foreword by D. G. Kendall.

[14] R. van de Weygaert. Fragmenting the Universe III. The construction and statistics of 3-D Voronoi tessellations. *Astron. Astrophys.*, 283:361–406, 1994.

[15] S. A. Zuyev. Estimates for distributions of the Voronoĭ polygon's geometric characteristics. *Random Structures Algorithms*, 3(2):149–162, 1992.

Precise formulas for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process. *

Pierre Calka[†]

Abstract

In this paper, we give an explicit integral expression for the joint distribution of the number and the respective positions of the sides of the typical cell $\mathcal{C}$ of a two-dimensional Poisson-Voronoi tessellation. We deduce from it precise formulas for the distributions of the principal geometric charateristics of $\mathcal{C}$ (area, perimeter, area of the fundamental domain). We also adapt the method to the Crofton cell and the empirical (or typical) cell of a Poisson line process.

1 Introduction and principal results.

1.1 The typical cell of a two-dimensional Poisson-Voronoi tessellation.

Consider Φ a homogeneous Poisson point process in $\mathbb{R}^2$, with the two-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^2; ||y-x|| \le ||y-x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [8] and Gilbert [4] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [7], telecommunications [1], astrophysics [17] and ecology [14]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [15] and Okabe et al. [13].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [12]. Consider the space $\mathcal{K}$ of convex compact

American Mathematical Society 2000 subject classifications. Primary 60D05; secondary 60G55.
Key words and phrases. Crofton cell, empirical cell, fundamental domain, Palm distribution, Poisson-Voronoi tessellation, stochastic geometry, typical cell.

[†] *Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr

sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [12]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B\cap\Phi} h(C(x)-x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^2; ||y|| \leq ||y-x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [12] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$. We will call a point y of Φ a *neighbor* of the origin if the bisecting line of the segment $[0, y]$ intersects the boundary of $C(0)$. Let us denote by $N_0(\mathcal{C})$ the number of sides (or equivalently vertices) of the typical cell $\mathcal{C}$. In [3], we provided an integral formula for the distribution function of $N_0(\mathcal{C})$. We extend the method to obtain the joint distribution of the respective positions of the k lines bounding $C(0)$ conditionally to the event $\{N_0(C(0)) = k\}$, $k \geq 3$.

Theorem 1 *(i) For every $k \geq 3$, we have*

$$\mathbf{P}\{N_0(\mathcal{C}) = k\} = \frac{(2\pi)^k}{k!}\int d\sigma_k(\delta_1, \cdots, \delta_k) \int \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})}\mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i)p_i dp_i \quad (1)$$

where σ_k is the (normalized) uniform measure on the simplex

$$\mathcal{S}_k = \{(\delta_1, \cdots, \delta_k) \in [0, 2\pi]; \sum_{i=1}^{k}\delta_i = 2\pi\}, \quad (2)$$

with

$$B = \{(p, q, r, \alpha, \beta) \in (\mathbb{R}_+)^3 \times (0, \pi)^2; p\sin(\beta) + r\sin(\alpha) \geq q\sin(\alpha+\beta)\}, \quad (3)$$

with for every $\delta \in (0, \pi)$, $p, q \geq 0$,

$$H(\delta, p, q) = \frac{1}{2\sin^2(\delta)}\left\{(p^2+q^2-2pq\cos(\delta))\frac{\delta}{2} + pq\sin(\delta) - \frac{p^2}{4}\sin(2\delta) - \frac{q^2}{4}\sin(2\delta)\right\}, \quad (4)$$

and with the conventions $p_0 = p_k$, $p_{k+1} = p_1$, and $\delta_0 = \delta_k$;
(ii) conditionally to $\{N_0(C(0)) = k\}$, let us denote by $(P_1, \Theta_1), \cdots, (P_k, \Theta_k)$ the polar coordinates of the consecutive neighbors of the origin in the trigonometric order.

The joint distribution of the vector

$$(P_1, \cdots, P_k, \Theta_2 - \Theta_1, \cdots, \Theta_k - \Theta_{k-1}, 2\pi + \Theta_1 - \Theta_k)$$

then has a density with respect to the measure

$$d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) = dp_1\cdots dp_k d\sigma_k(\delta_1,\cdots,\delta_k), \tag{5}$$

and its density φ_k is given by the following equality for every $p_1,\cdots,p_k \geq 0$, $(\delta_1,\cdots,\delta_k) \in \mathcal{S}_k$,

$$\begin{aligned}\varphi_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) &= \frac{1}{\mathbf{P}\{N_0(\mathcal{C})=k\}}\frac{(2\pi)^k}{k!} \\ &\qquad \prod_{i=1}^k p_i e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i).\end{aligned}$$

A table of numerical values for the distribution function of $N_0(\mathcal{C})$ has already been provided (see [3], table 1).

Let us denote by $\mathcal{F}(C(0))$ the fundamental domain associated to $C(0)$, i.e.

$$\mathcal{F}(C(0)) = \cup_{x\in C(0)} D(x,||x||),$$

where $D(y,r)$ is the disk centered at $y \in \mathbb{R}^2$ and of radius $r \geq 0$.

Theorem 1 provides an easy way to obtain the distribution of the area of $\mathcal{F}(C(0))$ conditionally to $\{N_0(C(0)) = k\}$, $k \geq 3$, and explicit integral formulas for the distribution of the area $V_2(\mathcal{C})$ and the perimeter $V_1(\mathcal{C})$ of $\mathcal{C}$.

Corollary 1 *Conditionally to the event $\{N_0(\mathcal{C}) = k\}$, $k \geq 3$,*
(i) the area $V_2(\mathcal{F}(C(0)))$ is Gamma distributed of parameters $(k,1)$;
(ii) the distribution of $V_2(\mathcal{C})$ is given by the following equality for every $t \geq 0$:

$$\mathbf{P}\{V_2(\mathcal{C}) \geq t | N_0(\mathcal{C}) = k\} = \int (\mathbf{1}_{C_t}\cdot\varphi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where

$$\begin{aligned}C_t = \{&(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) \in (\mathbb{R}_+)^k \times (0,\pi)^k; \\ &\frac{1}{8}\sum_{i=1}^k \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} p_i(p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i\sin(\delta_{i-1}+\delta_i)) \geq t\};\end{aligned} \tag{6}$$

(iii) the distribution of $V_1(\mathcal{C})$ is given by the following equality for every $t \geq 0$:

$$\mathbf{P}\{V_1(\mathcal{C}) \geq t | N_0(\mathcal{C}) = k\} = \int (\mathbf{1}_{E_t}\cdot\varphi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where

$$\begin{aligned}E_t = \{&(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) \in (\mathbb{R}_+)^k \times (0,\pi)^k; \\ &\frac{1}{2}\sum_{i=1}^k \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} (p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i\sin(\delta_{i-1}+\delta_i)) \geq t\}.\end{aligned}$$

Remark 1 The point (i) was already obtained by Zuyev [18] with a different method based on Russo's formula. The result can be easily extended to a d-dimensional Poisson-Voronoi tessellation, $d \geq 3$, in the following way: condtionally to the event {number of hyperfaces of $C(0) = k$}, $k \geq d+1$, the Lebesgue measure of the fundamental domain of $C(0)$ is Gamma distributed of parameters $(k,1)$.

1.2 The Crofton cell of a Poisson line process.

Let us now consider Φ' a Poisson point process in $\mathbb{R}^2$ of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_0^{2\pi} \mathbf{1}_A(r,u) d\theta dr, \quad A \in \mathcal{B}(\mathbb{R}^2).$$

Let us consider for all $x \in \mathbb{R}^2$, $H(x) = \{y \in \mathbb{R}^2; (y-x)\cdot x = 0\}$, ($x \cdot y$ being the usual scalar product). Then the set $\mathcal{H} = \{H(x); x \in \Phi\}$ is called a *Poisson line process* and divides the plane into convex polygons that constitute the so-called *two-dimensional Poissonian tessellation.* This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.

This random object was used for the first time by S. A. Goudsmit [6] and by R. E. Miles ([9], [10] and [11]). In particular, it provides a model for the fibrous structure of sheets of paper.

The origin is almost surely included in a unique cell C_0', called the *Crofton cell.* As in Theorem 1, we can get the joint distribution of the number of sides $N_0(C_0')$ of C_0' and the respective positions of its bounding lines.

Theorem 2 *(i) For every $k \geq 3$, we have*

$$\begin{aligned}\mathbf{P}\{N_0(C_0') = k\} &= \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \\ &\qquad \int \prod_{i=1}^k e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) dp_i \quad (7)\end{aligned}$$

(ii) conditionally to $\{N_0(C_0') = k\}$, let us denote by $(P_1', \Theta_1'), \cdots, (P_k', \Theta_k')$ the polar coordinates of the projections of the origin on the consecutive lines bounding C_0' in the trigonometric order.

The joint distribution of the vector

$$(P_1', \cdots, P_k', \Theta_2' - \Theta_1', \cdots, \Theta_k' - \Theta_{k-1}', 2\pi + \Theta_1' - \Theta_k')$$

then has a density with respect to the measure ν_k (defined by (5)) and its density φ_k' is given by the following equality for every $p_1, \cdots, p_k \geq 0$, $(\delta_1, \cdots, \delta_k) \in \mathcal{S}_k$,

$$\begin{aligned}\varphi_k'(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) &= \frac{1}{\mathbf{P}\{N_0(C_0') = k\}} \frac{(2\pi)^k}{k!} \\ &\qquad \prod_{i=1}^k e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i).\end{aligned}$$

As for the Voronoi case, the point (i) of Theorem 2 provides numerical values estimated by a Monte-Carlo procedure which are listed in Table 1.

We deduce from Theorem 2 the joint distributions of the couples $(N_0(C_0'), V_1(C_0'))$ and $(N_0(C_0'), V_2(C_0'))$.

k	3	4	5	6	7	8	9
$\mathbf{P}\{N_0(C_0') = k\}$	0.0767	0.3013	0.3415	0.1905	0.0682	0.0155	0.0052

Table 1: Numerical values for $\mathbf{P}\{N_0(C_0') = k\}$.

Corollary 2 *Conditionally to the event $\{N_0(C_0') = k\}$, $k \geq 3$,*
(i) the perimeter $V_1(C_0')$ is Gamma distributed of parameters $(k, 1)$;
(ii) the distribution of $V_2(C_0')$ is given by the following equality for every $t \geq 0$:

$$\begin{aligned}&\mathbf{P}\{V_2(C_0') \geq t | N_0(C_0') = k\} \\ &\qquad = \int (\mathbf{1}_{C_{t/4}} \cdot \varphi_k')(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) d\nu_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k),\end{aligned}$$

where the set $C_{t/4}$ is defined by the equality (6).

Remark 2 The point (i) can be extended to any d-dimensional Poissonian tessellation, $d \geq 3$, in the following way: conditionally to the event $\{$number of hyperfaces of $C_0' = k\}$, $k \geq d+1$, the mean width of C_0' is Gamma distributed of parameters $\left(k, \frac{\Gamma(d/2)}{\pi^{d/2}}\right)$.

1.3 The typical cell of a Poisson line process.

The notion of *typical* (or *empirical*) cell $\mathcal{C}'$ for the Poisson tessellation was first introduced by Miles [9], [10] through the convergence of ergodic means and has been recently reinterpreted by means of a Palm measure [2]. The typical cell $\mathcal{C}'$ is connected in law to the Crofton cell by the following equality (see for example [2]):

$$\mathbf{E}h(\mathcal{C}') = \frac{1}{\mathbf{E}(1/V_2(C_0'))} \mathbf{E}\left(\frac{h(C_0')}{V_2(C_0')}\right), \tag{8}$$

for all measurable and bounded function $h : \mathcal{K} \longrightarrow \mathbf{R}$ which is invariant by translation. Besides, it is well known [15] that

$$\mathbf{E}\{V_2(\mathcal{C}')\} = \left[\mathbf{E}\left(\frac{1}{V_2(C_0')}\right)\right]^{-1} = \frac{1}{\pi}. \tag{9}$$

Since Corollary 2 provides the joint distribution of the couple $(N_0(C_0'), V_2(C_0'))$, we can deduce from the equality (8) the law of the number of sides $N_0(\mathcal{C}')$ and also generalize all the results obtained for the Crofton cell.

Theorem 3 *(i) For every $k \geq 3$, we have*

$$\begin{aligned}\mathbf{P}\{N_0(\mathcal{C}') = k\} \quad = \quad \frac{(2\pi)^k}{\pi \cdot k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int \frac{\prod_{i=1}^k e^{-p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)}}{W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)} \\ \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) dp_1 \cdots dp_k, \end{aligned}\tag{10}$$

where

$$W_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)$$
$$=\frac{1}{2}\sum_{i=1}^{k}\frac{1}{\sin(\delta_{i-1})\sin(\delta_i)}p_i(p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})-p_i\sin(\delta_{i-1}+\delta_i));$$

(ii) Let

$$(Q_1,\cdots,Q_k,\Sigma_1,\cdots,\Sigma_k)\in(\mathbb{R}_+)^k\times\mathcal{S}_k$$

be a random vector which has a density ψ_k with respect to the measure ν_k (given by (5)) satisfying the following equality for every $p_1,\cdots,p_k\geq 0$, $(\delta_1,\cdots,\delta_k)\in\mathcal{S}_k$,

$$\psi_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k) \quad = \quad a_k\cdot\frac{\varphi'_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)}{W_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)}.$$

where $a_k=(\mathbf{P}\{N_0(C'_0)=k\}/(\pi\mathbf{P}\{N_0(\mathcal{C}')=k\}))$.

Let us consider a random angle Θ independent of the preceding vector and uniformly distributed on the circle. We denote by $X_1,X_2,\cdots,X_k$ the points of the plane of respective polar coordinates (Q_1,Θ), $(Q_2,\Theta+\Sigma_1),\cdots,(Q_k,\Theta+\Sigma_1+\cdots+\Sigma_{k-1})$. The typical cell $\mathcal{C}'$ then is equal in law to the convex polygon bounded by the lines $H(X_1),\cdots,H(X_k)$.

Numerical values for the distribution function of $N_0(\mathcal{C}')$ using the point (i) and a Monte-Carlo method are listed in Table 2. Let us remark that Miles [9] obtained that $\mathbf{P}\{N_0(\mathcal{C}')=3\}=2-\frac{\pi^2}{6}$ and Tanner [16] get the exact value for $\mathbf{P}\{N_0(\mathcal{C}')=4\}$.

As for the Crofton cell, we deduce from the preceding theorem a corollary about the joint distributions of the number of sides and the perimeter $V_1(\mathcal{C}')$ (resp. the area $V_2(\mathcal{C}')$) of the typical cell.

Corollary 3 *Conditionally to the event $\{N_0(\mathcal{C}')=k\}$, $k\geq 3$,*
(i) the perimeter $V_1(\mathcal{C}')$ is Gamma distributed of parameters $(k-2,1)$;
(ii) the distribution of $V_2(\mathcal{C}')$ is given by the following equality for every $k\geq 3$, $t\geq 0$:

$$\mathbf{P}\{V_2(\mathcal{C}')\geq t|N_0(\mathcal{C}')=k\}=\int(\mathbf{1}_{C_{t/4}}\cdot\psi_k)(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k)d\nu_k(p_1,\cdots,p_k,\delta_1,\cdots,\delta_k),$$

where the set $C_{t/4}$ is defined by the equality (6).

Remark 3 The point (i) was already obtained by R. E. Miles [9]. It can be extended to any d-dimensional Poissonian tessellation in the following way: conditionally to the event {number of hyperfaces of $\mathcal{C}'=k$}, $k\geq d+1$, the mean width of $\mathcal{C}'$ is Gamma distributed of parameters $\left(k-d,\frac{\Gamma(d/2)}{\pi^{d/2}}\right)$.

In the paper, we first prove the resuls relative to the Poisson-Voronoi tessellation and secondly the analogous facts for the Crofton cell of a Poisson line process. Let us remark that Theorem 3 and Corollary 3 are direct consequences of Theorem 2 and Corollary 2 combined with (8) and (9).

k	3	4	5	6	7	8	9
$\mathbf{P}\{N_0(\mathcal{C}') = k\}$	0.3554	0.3815	0.1873	0.0596	0.0129	0.0023	0.0004

Table 2: Numerical values for $\mathbf{P}\{N_0(\mathcal{C}') = k\}$.

2 Proofs of Theorem 1 and Corollary 1.

We use the same technique as in [3] based on Slivnyak's formula (see e.g. [12]).

For every $x \in \mathbb{R}^2$, let us denote by $L(x)$ (respectively $\mathcal{D}(x)$) the bisecting line of the segment $[0, x]$ (respectively the half-plane containing 0 delimited by $L(x)$).

We then define for all $k \geq 3$, and $x_1, \cdots, x_k \in \mathbb{R}^2$, the domain

$$\mathcal{D}(x_1, \cdots, x_k) = \cap_{i=1}^{k} \mathcal{D}(x_k).$$

Besides, we consider the set of $(\mathbb{R}^2)^k$

$$A_k = \{(x_1, \cdots, x_k) \in (\mathbb{R}^2)^k; \mathcal{D}(x_1, \cdots, x_k) \text{ is a convex polygon with } k \text{ sides}\}, \quad (11)$$

and for every $(x_1, \cdots, x_k) \in A_k$, the Lebesgue measure of the fundamental domain of $\mathcal{D}(x_1, \cdots, x_k)$, i.e.

$$V(x_1, \cdots, x_k) = V_2[\mathcal{F}(\mathcal{D}(x_1, \cdots, x_k))] = V_2\left[\cup_{x \in \mathcal{D}(x_1., \cdots, x_k)} D(x, ||x||)\right].$$

Let $\mathcal{N}_0$ be the set of all neighbors of the origin.

Proposition 1 *For every $k \geq 3$ and every bounded and measurable function $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ invariant by permutation, we have*

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} = \frac{1}{k!}\int h(x_1, \cdots, x_k) \exp\{-V(x_1, \cdots, x_k)\} \mathbf{1}_{A_k}(x_1, \cdots, x_k) dx_1 \cdots dx_k. \quad (12)$$

Proof. Let us decompose Ω over all possibilities for the set $\mathcal{N}_0$.

$$\begin{aligned}
&\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} \\
&= \mathbf{E}\left\{\sum_{\{x_1, \cdots, x_k\} \subset \Phi} h(x_1, \cdots, x_k) \mathbf{1}_{A_k}(x_1, \cdots, x_k) \mathbf{1}_{\{\mathcal{D}(x_1, \cdots, x_k) = C(0)\}}\right\} \\
&= \mathbf{E}\left\{\sum_{\{x_1, \cdots, x_k\} \subset \Phi} h(x_1, \cdots, x_k) \mathbf{1}_{A_k}(x_1, \cdots, x_k) \mathbf{1}_{\{L(y) \cap \mathcal{D}(x_1, \cdots, x_k) = \emptyset \ \forall y \in \Phi \setminus \{x_1, \cdots, x_k\}\}}\right\}.
\end{aligned}$$

Using Slivnyak's formula [12], we obtain

$$\begin{aligned}
&\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}} h(\mathcal{N}_0)\right\} \\
&= \frac{1}{k!}\int h(x_1, \cdots, x_k) \mathbf{1}_{A_k}(x_1, \cdots, x_k) \mathbf{E}\left(\mathbf{1}_{\{L(y) \cap \mathcal{D}(x_1, \cdots, x_k) = \emptyset \ \forall y \in \Phi\}}\right) dx_1 \cdots dx_k \\
&= \frac{1}{k!}\int h(x_1, \cdots, x_k) \mathbf{1}_{A_k}(x_1, \cdots, x_k) \\
&\qquad \mathbf{P}\{L(y) \cap \mathcal{D}(x_1, \cdots, x_k) = \emptyset \ \forall y \in \Phi\} dx_1 \cdots dx_k. \quad (13)
\end{aligned}$$

We can easily verify that for any $z \in \mathbb{R}^2$,

$$L(z) \cap \mathcal{D}(x_1, \cdots, x_k) \neq \emptyset \Longleftrightarrow z \in \cup_{x \in \mathcal{D}(x_1 \cdots, x_k)} D(x, ||x||),$$

From this remark and the Poissonian property of Φ, we get

$$\begin{aligned} \mathbf{P}\{L(y) \cap \mathcal{D}(x_1, \cdots, x_k) = \emptyset \, \forall y \in \Phi\} &= \mathbf{P}\{\Phi \cap \left[\cup_{x \in \mathcal{D}(x_1, \cdots, x_k)} D(x, ||x||)\right] = \emptyset\} \\ &= e^{-V(x_1, \cdots, x_k)}. \end{aligned} \tag{14}$$

Inserting the equality (14) in (13), we deduce Proposition 1.

□

We already expressed the set A_k analytically and calculated the area $V(x_1, \cdots, x_k)$ in function of the polar coordinates of $x_1, \cdots, x_k$ (see [3], lemmas 1 and 2). Let us denote by

$$(p_1, \theta_1), \cdots, (p_k, \theta_k) \in \mathbb{R}_+ \times [0, 2\pi),$$

the respective polar coordinates of $x_1, \cdots, x_k \in \mathbb{R}^2$. Supposing that $\theta_1, \cdots, \theta_k$ are in growing order, we define $\delta_i = \theta_{i+1} - \theta_i$, $1 \leq i \leq (k-1)$, and $\delta_k = 2\pi + \theta_1 - \theta_k$. We then have the two following results:

$$\mathbf{1}_{A_k}(x_1, \cdots, x_k) = \prod_{i=1}^{k} \mathbf{1}_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i), \tag{15}$$

where the set B is defined by (3), and for every $(x_1, \cdots, x_k) \in A_k$,

$$\begin{aligned} V(x_1, \cdots, x_k) = \sum_{i=1}^{k} \frac{1}{2\sin^2(\delta_i)} \Bigg\{ &(p_i^2 + p_{i+1}^2 - 2p_i p_{i+1} \cos(\delta_i)) \frac{\delta_i}{2} + p_i p_{i+1} \sin(\delta_i) \\ &- \frac{p_i^2}{4} \sin(2\delta_i) - \frac{p_{i+1}^2}{4} \sin(2\delta_i) \Bigg\}. \end{aligned} \tag{16}$$

Proof of Theorem 1. Using polar coordinates in the integral of the equality (12), we obtain for every $k \geq 3$,

$$\begin{aligned} &\mathbf{E}\{\mathbf{1}_{\{N_0(\mathcal{C})=k\}} h(\mathcal{N}_0)\} \\ &= \frac{1}{k!} \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} (h \cdot \mathbf{1}_{A_k})(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} \mathbf{1}_{\{0 \leq \theta_i \leq 2\pi\}} p_i dp_i d\theta_i \\ &= \int e^{-V(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k})} (h \cdot \mathbf{1}_{A_k})(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}) \\ &\qquad \mathbf{1}_{\{0 \leq \theta_1 \leq \cdots \leq \theta_k \leq 2\pi\}} \prod_{i=1}^{k} \mathbf{1}_{\{p_i \geq 0\}} p_i dp_i d\theta_i, \end{aligned} \tag{17}$$

where u_θ, $0 \leq \theta \leq 2\pi$, denotes the unit vector in the plane of rectangular coordinates $(\cos\theta, \sin\theta)$. Let us suppose that h is invariant under rotation, i.e. for all $\theta \in [0, 2\pi]$,

$$h(p_1 u_{\theta+\theta_1}, \cdots, p_k u_{\theta+\theta_k}) = h(p_1 u_{\theta_1}, \cdots, p_k u_{\theta_k}).$$

Inserting then the results (15) and (16) in (17), we deduce that

$$\begin{aligned}
&\mathbf{E}\{\mathbf{1}_{\{N_0(\mathcal{C})=k\}}h(\mathcal{N}_0)\}\\
&\quad=\int\Bigg[\int h(p_1u_0,p_2u_{\delta_1},\cdots,p_ku_{\delta_1+\cdots+\delta_{k-1}})\\
&\qquad\prod_{i=1}^{k}e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i)p_idp_i\Bigg]\mathbf{1}_{\{\delta_1+\cdots+\delta_{k-1}\le 2\pi\}}\delta_kd\delta_1\cdots d\delta_{k-1}\\
&\quad=\frac{(2\pi)^k}{k!}\int d\sigma_k(\delta_1,\cdots,\delta_k)\int h(p_1u_0,p_2u_{\delta_1},\cdots,p_ku_{\delta_1+\cdots+\delta_{k-1}})\\
&\qquad\qquad\prod_{i=1}^{k}e^{-H(\delta_i,p_i,p_{i+1})}\mathbf{1}_B(p_{i-1},p_i,p_{i+1},\delta_{i-1},\delta_i)p_idp_i,
\end{aligned}\tag{18}$$

where the function H is defined by the equality (4).

This last equality provides us the point (ii) of Theorem 1 and replacing h by $\mathbf{1}$, we obtain the point (i).

□

Proof of Corollary 1. Let us first notice that for every $(x_1,\cdots,x_k)\in A_k$,

$$V_2(\mathcal{D}(x_1,\cdots,x_k))=\frac{1}{8}\sum_{i=1}^{k}\frac{1}{\sin(\delta_{i-1})\sin(\delta_i)}p_i(p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})-p_i\sin(\delta_{i-1}+\delta_i)),\tag{19}$$

and

$$\begin{aligned}
&V_1(\mathcal{D}(x_1,\cdots,x_k))\\
&\qquad=\frac{1}{2}\sum_{i=1}^{k}\frac{1}{\sin(\delta_{i-1})\sin(\delta_i)}(p_{i-1}\sin(\delta_i)+p_{i+1}\sin(\delta_{i-1})-p_i\sin(\delta_{i-1}+\delta_i)) \quad (20)\\
&\qquad=\frac{1}{2}\sum_{i=1}^{k}p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)}+\frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right). \quad (21)
\end{aligned}$$

The point (ii) (resp. (iii)) then is easily obtained by applying the equality (18) to

$$h(x_1,\cdots,x_k)=\mathbf{1}_{\{V_2(\mathcal{D}(x_1,\cdots,x_k))\ge t\}}$$

(resp. $h(x_1,\cdots,x_k)=\mathbf{1}_{\{V_1(\mathcal{D}(x_1,\cdots,x_k))\ge t\}}$). As for point (i), let us apply the equality (12) to

$$h(x_1,\cdots,x_k)=e^{-\lambda V(x_1,\cdots,x_k)},\quad \lambda\ge 0.$$

Let us notice that if $\mathcal{N}_0=\{x_1,\cdots,x_k\}$, we have $V(x_1,\cdots,x_k)=V_2(\mathcal{F}(C(0)))$.

Consequently, we obtain

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}}e^{-\lambda V_2(\mathcal{F}(C(0)))}\right\}=\frac{1}{k!}\int e^{-(\lambda+1)V(x_1,\cdots,x_k)}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k.$$

We take the change of variables $x'_i = \sqrt{\lambda+1}x_i$, $1 \le i \le k$, to deduce that

$$\begin{aligned}\mathbf{E}\left\{\mathbf{1}_{\{N_0(C(0))=k\}}e^{-\lambda V_2(\mathcal{F}(C(0)))}\right\} &= \frac{1}{(\lambda+1)^k}\cdot\frac{1}{k!}\int e^{-V(x_1,\cdots,x_k)}\mathbf{1}_{A_k}(x_1,\cdots,x_k)dx_1\cdots dx_k \\ &= \mathbf{P}\{N_0(C(0)=k\}\frac{1}{(\lambda+1)^k}.\end{aligned}$$

So conditionally to the event $\{N_0(C(0)) = k\}$, the Laplace transform of the distribution of $V_2(\mathcal{F}(C(0)))$ is exactly $(\lambda+1)^{-k}$, $\lambda \ge 0$, i.e. $V_2(\mathcal{F}(C(0)))$ is Gamma distributed with parameters $(k, 1)$.

□

3 Proofs of Theorem 2 and Corollary 2.

For all $x \in \mathbb{R}^2$, let us define $\mathcal{D}'(x)$ as the half-plane containing the origin delimited by the line $H(x)$. We then denote for every $x_1, \cdots, x_k \in \mathbb{R}^2$,

$$\mathcal{D}'(x_1,\cdots,x_k) = \mathcal{D}'(x_1)\cap\cdots\cap\mathcal{D}'(x_k) = \mathcal{D}(2x_1,\cdots,2x_k).$$

Let $\mathcal{N}'_0$ be the (random) set of all points $x \in \Phi'$ such that $H(x)$ intersects the boundary of the Crofton cell C'_0.

Proposition 2 *For every $k \ge 3$ and every bounded and measurable function $h : \mathbb{R}^k \longrightarrow \mathbb{R}$ invariant by permutation, we have*

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C'_0)=k\}}h(\mathcal{N}'_0)\right\} = \frac{1}{k!}\int (h\cdot\mathbf{1}_{A_k})(x_1,\cdots,x_k)\exp\{-V_1(\mathcal{D}'(x_1,\cdots,x_k))\}dx_1\cdots dx_k. \tag{22}$$

Proof. As for Proposition 1, we apply Slivnyak's formula to obtain

$$\begin{aligned}\mathbf{E}\left\{\mathbf{1}_{\{N_0(C'_0)=k\}}h(\mathcal{N}'_0)\right\} &= \frac{1}{k!}\int h(x_1,\cdots,x_k)\mathbf{1}_{A_k}(x_1,\cdots,x_k) \\ &\qquad \mathbf{P}\{H(y)\cap\mathcal{D}'(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi'\}dx_1\cdots dx_k. \quad (23)\end{aligned}$$

We can easily verify (see e.g. [5]) that

$$\begin{aligned}\mathbf{P}\{H(y)\cap\mathcal{D}'(x_1,\cdots,x_k)=\emptyset\ \forall y\in\Phi'\} &= \mathbf{P}\{\mathcal{D}'(x_1,\cdots,x_k)\subset C'_0\} \\ &= \exp\{-V_1(\mathcal{D}'(x_1,\cdots,x_k))\}. \quad (24)\end{aligned}$$

Inserting the equality (24) in (23), we deduce Proposition 2.

□

Proofs of Theorem 2 and Corollary 2. Let us recall that

$$V_1(\mathcal{D}'(x_1,\cdots,x_k)) = \sum_{i=1}^k p_i\left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right), \tag{25}$$

and

$$V_2(\mathcal{D}'(x_1, \cdots, x_k)) = \frac{1}{2}\sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1})\sin(\delta_i)} p_i(p_{i-1}\sin(\delta_i) + p_{i+1}\sin(\delta_{i-1}) - p_i \sin(\delta_{i-1}+\delta_i)). \tag{26}$$

It then suffices to insert in (22) the results (15) and (25) to obtain the two points of Theorem 2.

The proof of Corollary 2 is also analogous to the Voronoi case. In particular, point (i) is deduced from a calculation of the Laplace transform of the distribution of the perimeter of C_0' conditioned by the event $\{N_0(C_0') = k\}$, $k \geq 3$:

$$\mathbf{E}\left\{\mathbf{1}_{\{N_0(C_0')=k\}} e^{-\lambda V_1(C_0')}\right\} = \mathbf{P}\{N_0(C_0') = k\} \cdot \frac{1}{(\lambda+1)^k}, \quad \lambda \geq 0.$$

□

References

[1] F. Baccelli and B. Błaszczyszyn. On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Adv. in Appl. Probab.*, 33(2):293–323, 2001.

[2] P. Calka. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6):557–562, 2001.

[3] P. Calka. The explicit expression of the distribution of the number of sides of the typical Poisson-Voronoi cell. Preprint of LaPCS, 02-02, 2002.

[4] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33:958–972, 1962.

[5] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[6] S. Goudsmit. Random distribution of lines in a plane. *Rev. Modern Phys.*, 17:321–322, 1945.

[7] S. Kumar and R. N. Singh. Thermal conductivity of polycristalline materials. *J. of the Amer. Cer. Soc.*, 78(3):728–736, 1995.

[8] J. L. Meijering. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.*, 8, 1953.

[9] R. E. Miles. Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci. U.S.A.*, 52:901–907, 1964.

[10] R. E. Miles. Random polygons determined by random lines in a plane. II. *Proc. Nat. Acad. Sci. U.S.A.*, 52:1157–1160, 1964.

[11] R. E. Miles. The various aggregates of random polygons determined by random lines in a plane. *Advances in Math.*, 10:256–290, 1973.

[12] J. Møller. *Lectures on random Voronoĭ tessellations*. Springer-Verlag, New York, 1994.

[13] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams*. John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[14] E. Pielou. *Mathematical ecology*. Wiley-Interscience, New-York, 1977.

[15] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications*. John Wiley & Sons Ltd., Chichester, 1987. With a foreword by D. G. Kendall.

[16] J. C. Tanner. Polygons formed by random lines in a plane: some further results. *J. Appl. Probab.*, 20(4):778–787, 1983.

[17] R. van de Weygaert. Fragmenting the Universe III. The construction and statistics of 3-D Voronoi tessellations. *Astron. Astrophys.*, 283:361–406, 1994.

[18] S. A. Zuyev. Estimates for distributions of the Voronoĭ polygon's geometric characteristics. *Random Structures Algorithms*, 3(2):149–162, 1992.

The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *

Pierre Calka†

Abstract

Denote by R_m (respectively R_M) the radius of the largest (respectively smallest) disk centered at a typical particle of the two-dimensional Poisson-Voronoi tessellation and included within (respectively containing) the polygonal cell associated with that particle. In this article, we obtain the joint distribution of R_m and R_M. This result is derived from the covering properties of the circle due to Stevens, Siegel and Holst. The same method works for studying the Crofton cell associated to the Poisson line process in the plane. The computation of the conditional probabilities $\mathbf{P}\{R_M \geq r+s | R_m = r\}$ reveals the circular property of the Poisson-Voronoi typical cells (as well as the Crofton cells) having a "large" in-disk.

Introduction and presentation of results.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^2$, with the 2-dimensional Lebesgue measure V_2 for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^d; ||y-x|| \leq ||y-x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polygons) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^2$. Introduced by Meijering [12] and Gilbert [4] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [11], telecommunications [1], astrophysics [26] and ecology [20]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [25] and Okabe et al. [18].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [16]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set

**American Mathematical Society 1991 subject classifications.* Primary 60D05; secondary 60G55.
.*Key words and phrases.* Coverage of the circle, Crofton cell, Palm distribution, Poisson line process, Poisson-Voronoi tessellation, stochastic geometry, typical cell.

†*Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr

$B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [16]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_2(B)}\mathbf{E}\sum_{x \in B\cap\Phi} h(C(x) - x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^2; ||y|| \le ||y - x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [16] that $C(0)$ and $\mathcal{C}$ are equal in law. From now on, we will use $C(0)$ as a realization of the typical cell $\mathcal{C}$.

The explicit distributions of the main geometrical characteristics of the typical cell are mainly unknown. For example, we do not have any precise idea of the asymptotic behaviour of the distribution function of the area of $\mathcal{C}$ and the best estimation up to now was obtained by Gilbert [4] in 1961 (see also [18]):

$$e^{-4t} \le \mathbf{P}\{V_2(\mathcal{C}) \ge t\} \le \frac{t-1}{e^{t-1}-1}, \quad t > 0. \tag{1}$$

Nevertheless, the law of the radius R_m of the largest ball centered at the origin and contained in $C(0)$ can be obtained easily. Indeed:

$$\begin{aligned} \mathbf{P}\{R_m \ge r\} &= \mathbf{P}\{D(r) \subset C(0)\} \\ &= \mathbf{P}\{\Phi \cap D(2r) = \emptyset\} = e^{-4\pi r^2}, \quad r > 0, \end{aligned}$$

where $D(r)$, $r > 0$, denotes the closed disk centered at the origin of radius r.

It is more difficult to determine the law of the radius R_M of the smallest disk centered at the origin containing $C(0)$. This problem was investigated by Foss and Zuyev [3] in the framework of a mathematical modelization of a telecommunications network. They obtained the following upper bound:

$$\mathbf{P}\{R_M \ge r\} \le 7e^{-\mu r^2}, \quad r > 0, \tag{2}$$

where $\mu = 2(\sin(\pi/14)\cos(5\pi/14) + \pi/7) \approx 1.09$.

In this work, we obtain the exact distribution of R_M.

Theorem 1 *The law of R_M is given by the following equality*

$$\mathbf{P}\{R_M \ge r\} = e^{-4\pi r^2}\left(1 - \sum_{k\ge1}\frac{(-4\pi r^2)^k}{k!}\xi_k\right), \quad r > 0, \tag{3}$$

with

$$\xi_k = \int \left[\prod_{i=1}^k F(u_i)\right] e^{4\pi r^2 \sum_{i=1}^k \int_0^{u_i} F(t)dt} d\sigma_k(u), \quad k \ge 1,$$

and

$$F(t) \quad = \quad \begin{cases} \sin^2(\pi t) & \text{if } 0 \le t \le 1/2 \\ 1 & \text{if } t \ge 1/2, \end{cases} \tag{4}$$

where σ_k *denotes the (normalized) area measure of the simplex*

$$\{u = (u_1, \cdots, u_k) \in [0,1]^k; \sum_{i=1}^{k} u_i = 1\}.$$

The proof of Theorem 1 relies on the observation that $\mathbf{P}\{R_M \ge r\}$ can be expressed in terms of probabilities of coverage of a circle by random independent and identically distributed arcs. More precisely, for any probability measure ν on $[0,1]$, let us denote by $P(\nu, n)$ the probability of the coverage of a circle of circumference one by n open random arcs $\mathcal{A}_i$, $1 \le i \le n$ such that:

(i) the lengths $0 \le L_i \le 1$, $1 \le i \le n$, of the arcs are independent and identically distributed random variables of law ν;

(ii) The centers C_i, $1 \le i \le n$, of these arcs are independent and uniformly distributed (on the unit circle) random variables;

(iii) The sequences $\{L_i; i \ge 1\}$ and $\{C_i; i \ge 1\}$ are independent.

We show that

Theorem 2 *For all* $r \ge 0$*,*

$$\mathbf{P}\{R_M \ge r\} = \sum_{n \ge 0} e^{-4\pi r^2} \frac{(4\pi r^2)^n}{n!} (1 - P(\nu_0, n)), \tag{5}$$

where $\nu_0(dt) = \pi \sin(2\pi t) \mathbf{1}_{[0,1/2]}(t) dt$*.*

The probabilities $P(\nu, n)$ (see formula (18)) were explicitely calculated by Siegel and Holst [23]. By inserting their expressions in (5), we obtain Theorem 1.

Using *Matlab*, we obtain precise estimates for $P(\nu_0, n)$, $n \ge 0$ that we insert in (5). It provides us numerical values for the distribution function of R_M that are listed in Table 1.

Besides, we deduce from Theorem 2 theoretical lower and upper bounds for $\mathbf{P}\{R_M \ge r\}$ that improve significantly the latest result (2) due to Foss and Zuyev:

Theorem 3 *For all* $r > 0$*, we have*

$$\begin{aligned} &2\pi r^2 e^{-\pi r^2} \left(1 + \frac{1}{2\pi r^2} e^{-\pi r^2}\right) \\ &\qquad \le \mathbf{P}\{R_M \ge r\} \\ &\qquad\qquad \le 2\pi r^2 e^{-\pi r^2} \left(2 - 2\pi r^2 e^{-\pi r^2} + \frac{\pi^2 r^4}{3} e^{-2\pi r^2} + \frac{1}{2\pi r^2} e^{-3\pi r^2}\right). \end{aligned}$$

In particular, for $r \ge \alpha \approx 0.337$*,*

$$2\pi r^2 e^{-\pi r^2} \le \mathbf{P}\{R_M \ge r\} \le 4\pi r^2 e^{-\pi r^2}. \tag{6}$$

r	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mathbf{P}\{R_M \geq r\}$	1	0.999	0.995	0.983	0.946	0.874	0.758	0.604	0.441
r	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\mathbf{P}\{R_M \geq r\}$	0.292	0.177	0.098	0.050	0.023	0.010	0.004	0.001	0

Table 1: Numerical values for $\mathbf{P}\{R_M \geq r\}$.

These estimations, particularly essential when r is large, are difficult to obtain. In order to do it, we use a conjecture of Siegel [22] which we deduce from a non-trivial result proved by Huffer and Shepp [9].

Let us notice that Theorem 3 provides an upper bound for the distribution function of the area of $\mathcal{C}$ which is better than Gilbert's one (1) for $0 < t \leq t^* \approx 1.043$ and worse for $t \geq t^*$.

By the same method we obtain the conditional distributions

$$\mathbf{P}\{R_M \geq t | R_m = r\}, \quad r \geq 0, t > 0$$

as well as the corresponding asymptotic estimations.

Theorem 4 *For all $r, s > 0$,*

$$\begin{aligned} \mathbf{P}\{R_M \geq r+s | R_m = r\} &= e^{-4\pi(s^2+2rs)a_{r,s}} \\ &\quad + e^{-4\pi(s^2+2rs)} \left(1 + \sum_{k\geq 1} (-1)^k \frac{(4\pi(s^2+2rs))^k}{k!} \xi_k(r,s)\right), \end{aligned} \tag{7}$$

where, for all $k \geq 1$,

$$\begin{aligned} \xi_k(r,s) &= \int \mathbf{1}_{\{u_1 \geq l_{r,s}\}} \left[\prod_{i=2}^{k+1} F_{r,s}(u_i)\right] e^{4\pi(s^2+2rs)\sum_{i=1}^{k+1}\int_0^{u_i} F_{r,s}(t)dt} d\sigma_{k+1}(u) \\ &\quad - \int \left[\prod_{i=1}^{k} F_{r,s}(u_i)\right] e^{4\pi(s^2+2rs)\sum_{i=1}^{k}\int_0^{u_i} F_{r,s}(t)dt} \left[\sum_{i=1}^{k} (u_i - l_{r,s})_+\right] d\sigma_k(u), \end{aligned}$$

with

$$l_{r,s} = \arccos(r/(r+s))/\pi, \tag{8}$$

$$F_{r,s}(t) = \nu_{r,s}([0,t]) = \begin{cases} \frac{(r+s)^2}{2rs+s^2}\sin^2(\pi t) & \text{if } 0 \leq t \leq l_{r,s} \\ 1 & \text{if } t \geq l_{r,s}, \end{cases} \tag{9}$$

and

$$\begin{aligned} a_{r,s} &= \int_0^1 t\, d\nu_{r,s}(t) \\ &= \frac{1}{2\sqrt{2}\pi}\sqrt{\frac{r}{s}}(1+\frac{s}{2r})^{-1/2} + \frac{1}{\pi}\arccos(1-\frac{s}{s+r})(1+\frac{s}{2r})^{-1}(-\frac{r}{4s}+\frac{1}{2}+\frac{s}{4r}). \end{aligned} \tag{10}$$

Theorem 5 *For all $0 < c < 8/(3\sqrt{2})$ and all fixed $-1 < \alpha < 1/3$,*

$$\mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} = O(e^{-cr^{\frac{1}{2}(1-3\alpha)}}), \quad \text{when } r \to +\infty. \tag{11}$$

The asymptotic result (11) follows from (7) and an inequality proved by Shepp (see [21]). It means that the boundary of the cells such that the in-disk (centered at the nucleus associated to the cell) has a "large radius" r, is included in the annulus $A(r, r + 1/r^\alpha)$ (with probability close to one). We observe, expressed in a different form, the circular property of the large cells of the two-dimensional Poisson-Voronoi tessellation that we already noticed in [7].

Besides, we can adapt the procedure to study the radius R'_M of the smallest disk centered at the origin containing the Crofton cell of the Poisson line process in the plane, of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_0^{2\pi} \mathbf{1}_A(\rho, \theta) d\theta d\rho, \quad A \in \mathcal{B}(\mathbb{R}^2).$$

Theorem 6 *The law of R'_M is given by the equality*

$$\mathbf{P}\{R'_M \geq r\} = e^{-2\pi r} \left(1 - \sum_{k \geq 1} \frac{(-2\pi r)^k}{k!} \zeta_k \right), \quad r > 0, \tag{12}$$

where for any $k \geq 1$,

$$\zeta_k = \int \left[\prod_{i=1}^k G(u_i) \right] e^{4\pi r^2 \sum_{i=1}^k \int_0^{u_i} G(t) dt} d\sigma_k(u),$$

and

$$G(t) = \begin{cases} 1 - \cos(\pi t) & \text{if } 0 \leq t \leq 1/2 \\ 1 & \text{if } t \geq 1/2. \end{cases} \tag{13}$$

Theorem 7 *We have*

$$\mathbf{P}\{R'_M \geq r\} = \sum_{n \geq 0} e^{-2\pi r} \frac{(2\pi r)^n}{n!} (1 - P(\nu'_0, n)),$$

where $\nu'_0(dt) = \pi \sin(\pi t) \mathbf{1}_{[0,1/2]}(t) dt$.

Theorem 8 *We have*

$$2\pi r e^{-2r} \left(\cos 1 + \frac{e^{-2(\pi \cos 1 - 1)r}}{2\pi r} \right) \leq \mathbf{P}\{R'_M \geq r\}$$
$$\leq 2\pi r e^{-2r} \left(1 - (\pi - 2) r e^{-2r} + \frac{2}{3} (\pi - 3)^2 r^2 e^{-4r} + \frac{e^{-2(\pi - 1)r}}{2\pi r} \right).$$

Denoting by R'_m the radius of the largest disk centered at the origin and contained in the cell, we have:

Theorem 9 *For all* $r, s > 0$,

$$\mathbf{P}\{R'_M \geq r+s | R'_m = r\} = e^{-2\pi s b_{r,s}} + e^{-2\pi s}\left(1 + \sum_{k\geq 1}(-1)^k \frac{(2\pi s)^k}{k!}\zeta_k(r,s)\right),$$

where for any $k \geq 1$,

$$\begin{aligned}\zeta_k(r,s) &= \int \mathbf{1}_{\{u_1 \geq l_{r,s}\}}\left[\prod_{i=2}^{k+1} G_{r,s}(u_i)\right] e^{2\pi s \sum_{i=1}^{k+1}\int_0^{u_i} G_{r,s}(t)dt} d\sigma_{k+1}(u) \\ &\quad - \int \left[\prod_{i=1}^{k} G_{r,s}(u_i)\right] e^{2\pi s \sum_{i=1}^{k}\int_0^{u_i} G_{r,s}(t)dt}\left[\sum_{i=1}^{k}(u_i - l_{r,s})_+\right] d\sigma_k(u),\end{aligned}$$

with

$$G_{r,s}(t) = \nu'_{r,s}([0,t]) = \begin{cases} \frac{r+s}{s}(1-\cos(\pi t)) & \text{if } 0 \leq t \leq l_{r,s} \\ 1 & \text{if } t \geq l_{r,s}, \end{cases} \tag{14}$$

and

$$\begin{aligned} b_{r,s} &= \int_0^1 t \, d\nu'_{r,s}(t) \\ &= \frac{\sqrt{2}}{\pi}\sqrt{\frac{r}{s}}(1+\frac{s}{2r})^{1/2} - \frac{r}{\pi s}\arccos(1-\frac{s}{s+r}). \end{aligned} \tag{15}$$

Theorem 10 *For all* $0 < c < 8/(3\sqrt{2})$ *and all fixed* $1/3 < \alpha < 1$,

$$\mathbf{P}\{R'_M \geq r + r^\alpha | R'_m = r\} = O(e^{-cr^{\frac{1}{2}(3\alpha-1)}}) \quad \text{when } r \to +\infty.$$

This paper is structured as follows. We prove first Theorem 2 which connects the distribution of R_M to the coverage probabilities. From this we deduce Theorem 1. Then we prove the conjecture of Siegel that the probability of coverage of the circle is an increasing function of the concentration (about the mean) of the distribution of the arc lengths. This result (which seems to be unknown) is derived quite easily from a comparison lemma of Huffer and Shepp. It provides us Theorem 3. Then we apply the same method to determine the conditional distributions $\mathbf{P}\{R_M \geq r+s | R_m = r\}$, $r, s > 0$ (Theorems 4 and 5). Finally we conclude this article by using the same arguments in order to obtain similar results in the case of the Crofton cell of the Poisson line process in the plane (Theorems 6 to 10).

1 Proofs of Theorems 1 and 2.

The probability of coverage of the circle by arcs of constant length equal to $0 \leq a \leq 1$ (corresponding to the choice $\nu = \delta_a$) was obtained by Stevens [23]. A proof of the following theorem can be found in [24].

Theorem 11 (Stevens, 1939) *For all $n \geq 1$, we have*

$$P(\delta_a, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (1 - ka)_+^{n-1}. \tag{16}$$

In particular, we deduce easily from Theorem 11 the following corollary.

Corollary 1 *For all $p \in [0, 1]$ and all $n \geq 1$, we have*

$$P((1-p)\delta_0 + p\delta_a, n) = 1-(1-p)^n + \sum_{k=1}^{n} (-1)^k \binom{n}{k} p^k (1-ka)_+^{k-1} [1-p+p(1-ka)_+]^{n-k}. \tag{17}$$

The formula (16) was extended to the general case by Siegel and Holst [23] under the form:

Theorem 12 (Siegel, Holst, 1982) *For any probability measure ν on $[0, 1]$ with F_ν for distribution function, we have*

$$P(\nu, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int \left[\prod_{i=1}^{k} F_\nu(u_i) \right] \left[\sum_{i=1}^{k} \int_0^{u_i} F_\nu(t) dt \right]^{n-k} d\sigma_k(u), \quad n \in \mathbb{N}^*, \tag{18}$$

where σ_k, $k \geq 1$, is the (normalized) uniform measure of the simplex

$$\{u = (u_1, \cdots, u_k); \sum_{i=1}^{k} u_i = 1\}$$

.

The formula (3) giving the law of R_M is derived directly from (5) and (18). It remains to prove Theorem 2.

Let us fix $r > 0$ and notice first that the convexity of $C(0)$ implies that

$$R_M \geq r \Longleftrightarrow \text{ there exists } x \in C(0) \text{ such that } ||x|| = r.$$

From the definition of the cell $C(0)$, we deduce the identity:

$$\begin{aligned} \{R_M \geq r\} &= \{\exists x \in C(0); ||x|| = r\} \\ &= \{\exists x; ||x|| = r \text{ and } ||x - y|| \geq r \ \ \forall y \in \Phi\} \\ &= \{\exists x; ||x|| = r \text{ and } ||x - y|| \geq r \ \ \forall y \in \Phi \cap D(2r)\}, \end{aligned} \tag{19}$$

Let us define for all $x \in D(2r)$,

$$\mathcal{A}(x) = \{y; ||y|| = r \text{ and } ||y - x|| < r\}.$$

The sets $\mathcal{A}(x)$, $x \in D(2r)$, are open arcs of the circle of radius $r > 0$ (see Figure 2). From (19) we get:

$$\{R_M \geq r\} = \{\exists x; ||x|| = r \text{ and } x \notin \cup_{y \in \Phi \cap D(2r)} \mathcal{A}(y)\}. \tag{20}$$

Besides, let us recall that

$$\Phi \cap D(2r) = \{X_n; 1 \leq n \leq N\},$$

where:

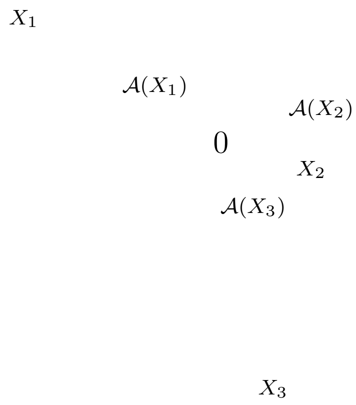

Figure 1: Covering the circle with the arcs $\mathcal{A}(X_i)$, $1 \leq i \leq n$.

(i') $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables, taking values in $D(2r)$, of law:

$$X_1(\mathbf{P}) = \frac{1}{4\pi r^2}\mathbf{1}_{D(2r)}(x)dx;$$

(ii') N is a Poisson variable of mean $\mathbf{E}N = 4\pi r^2$ and independent of the sequence $\{X_n; n \geq 1\}$.

Let us note

$$\mathcal{A}_i = \frac{1}{2\pi r}\mathcal{A}(X_i), \quad i \geq 1.$$

Then by an elementary geometrical argument, we deduce from (i') that for all $n \geq 1$, the sequence $\{\mathcal{A}_i; 1 \leq i \leq n\}$, satisfies the conditions (i)-(iii) with

$$L_i = \frac{1}{\pi}\arccos\left(\frac{||X_i||}{2r}\right), \quad i \geq 1,$$

which corresponds to the fact that the law ν_0 of the arc lengths is

$$\nu_0(dt) = \pi\sin(2\pi t)\mathbf{1}_{[0,\frac{1}{2}]}(t)dt.$$

Finally applying the property (ii') we obtain with (20):

$$\begin{aligned}\mathbf{P}\{R_M \geq r\} &= \mathbf{P}\{N=0\} + \sum_{n\geq 1}\mathbf{P}\{N=n\} \times \mathbf{P}\{\exists x; ||x|| = r \text{ and } x \notin \cup_{1\leq i\leq n}\mathcal{A}(X_i)\} \\ &= e^{-4\pi r^2}\left(1 + \sum_{n\geq 1}\frac{(4\pi r^2)^n}{n!}(1 - P(\nu_0, n))\right).\end{aligned}$$

This is the required result.

$\square$

2 A result of comparison for probabilities of coverage.

A. F. Siegel introduced [22] the following notion of comparison of probability distributions.

Definition 1 (Siegel, 1978) *Consider ν_1 and ν_2 two probability distributions on $[0,1]$ with common expectation*

$$\int_0^1 t\, d\nu_1(t) = \int_0^1 t\, d\nu_2(t) = e \in [0,1]. \tag{21}$$

ν_1 is said to be more concentrated (about the mean) than ν_2 if

$$\begin{cases} \nu_1([0,t]) \leq \nu_2([0,t]) & \text{for } t < e \\ \nu_1([0,t]) \geq \nu_2([0,t]) & \text{for } t \geq e \end{cases}$$

In particular, if ν_1 is more concentrated (about the mean) than ν_2, then ν_1 is *less dangerous than* ν_2, which is written in modern notation as $\nu_1 \leq_D \nu_2$ (see [17], p. 23).

Using simulation observations, A. F. Siegel conjectured [22] that

Theorem 13 *If ν_1 is more concentrated than ν_2, then we have*

$$P(\nu_1, n) \leq P(\nu_2, n) \quad \forall n \geq 1. \tag{22}$$

We are going to prove Siegel's conjecture. Let us remark first that

$$P(\nu_1, n) = \int \overline{P}(l_1, \cdots, l_n) d\nu_1(l_1) \cdots d\nu_1(l_n), \tag{23}$$

where $\overline{P}(l_1, \cdots, l_n)$, $l_1, \cdots, l_n \in [0,1]$, denotes the probability that the circle of circumference one is covered by n arcs of lengths respectively equal to $l_1, \cdots, l_n$, where the centers of the arcs are independent and uniformly distributed (on the circle) random variables.

Moreover, Huffer and Shepp [9] proved the following non-trivial result.

Theorem 14 (Huffer, Shepp, 1987) *The function*

$$(l_1, \cdots, l_n) \longmapsto \overline{P}(l_1, \cdots, l_n)$$

is convex in each argument when the others are held fixed.

Besides, it is well known (see [17], pages 16-17, 23) that the comparison $\nu_1 \leq_D \nu_2$ associated to (21) implies $\nu_1 \leq_{cx} \nu_2$, i.e. comparison in the convex order of distributions, which by definition ensures that for any convex function f on $[0,1]$, we have

$$\int f(t) d\nu_1(t) \leq \int f(t) d\nu_2(t). \tag{24}$$

Then applying Theorem 14 and (24), we obtain by successive iterations

$$\begin{aligned}
&\int \left[\int \overline{P}(l_1, l_2, \cdots, l_n) d\nu_1(l_2) \cdots d\nu_1(l_n) \right] d\nu_1(l_1) \\
&\qquad \leq \int \left[\int \overline{P}(l_1, l_2, \cdots, l_n) d\nu_1(l_2) \cdots d\nu_1(l_n) \right] d\nu_2(l_1) \\
&\qquad = \int \left[\int \overline{P}(l_1, l_2, \cdots, l_n) d\nu_2(l_1) d\nu_1(l_3) \cdots d\nu_1(l_n) \right] d\nu_1(l_2) \\
&\qquad \leq \int \left[\int \overline{P}(l_1, l_2, \cdots, l_n) d\nu_2(l_1) d\nu_1(l_3) \cdots d\nu_1(l_n) \right] d\nu_2(l_2) \\
&\qquad \vdots \\
&\qquad \leq \int \overline{P}(l_1, l_2, \cdots, l_n) d\nu_2(l_1) \cdots d\nu_2(l_n).
\end{aligned}$$

Consequently, using (23), we get

$$P(\nu_1, n) \leq P(\nu_2, n), \quad n \geq 1.$$

□

3 Proof of Theorem 3.

The upper and lower bounds on the distribution function of R_M given by Theorem 3 may be obtained easily from the preceding comparison result. Actually, we have clearly

$$\int t \, d\nu_0(t) = \pi \int_0^{1/2} t \sin(2\pi t) dt = \frac{1}{4},$$

and besides $\nu_0([0, 1/4]) = 1/2$. So

Lemma 1 *(i) The measure $\delta_{1/4}$ is more concentrated than ν_0;*

(ii) The measure ν_0 is more concentrated than $\frac{1}{2}(\delta_0 + \delta_{1/2})$.

Moreover, Stevens's formula (Theorem 11) and Corollary 1 provide the following expressions.

Lemma 2 *We have for all $n \geq 1$,*

$$1 - P(\delta_{1/4}, n) = n \left(\frac{3}{4}\right)^{n-1} - \binom{n}{2} \left(\frac{1}{2}\right)^{n-1} + \binom{n}{3} \left(\frac{1}{4}\right)^{n-1} \tag{25}$$

$$1 - P(1/2(\delta_0 + \delta_{1/2}), n) = 2^{-n} + \frac{n}{2} \left(\frac{3}{4}\right)^{n-1}. \tag{26}$$

Consequently, it suffices to apply Theorems 2 and 13 as well as Lemmas 1 and 2.

□

4 Proofs of Theorems 4 and 5.

Proof of Theorem 4. Notice first the following identity

$$\{\Phi | R_m = r\} \stackrel{law}{=} \Phi_r \cup \{X_0\},$$

where

(i) Φ_r is a Poisson planar point process of intensity measure $\mathbf{1}_{D(2r)^c} dx$

(ii) X_0 is a random variable uniformly distributed on the circle centered at the origin of radius $2r$, and independent of Φ_r.

So we can apply word for word the arguments described in the proof of Theorem 1 by replacing Φ by the point process $\Phi_r \cup \{X_0\}$. We obtain

$$\begin{aligned}
\mathbf{P}\{R_M \geq r + s | R_m = r\} & \\
&= \mathbf{P}\{\exists x; ||x|| = r + s \text{ and } x \notin \cup_{y \in \Phi_r \cup \{X_0\}} \mathcal{A}(y)\} \\
&= \sum_{n \geq 0} \mathbf{P}\{N = n\} \times \mathbf{P}\{\exists x; ||x|| = r + s \text{ and } x \notin \cup_{0 \leq i \leq n} \mathcal{A}(X_i)\}, \quad (27)
\end{aligned}$$

where

(i) $\{X_n, n \geq 1\}$ is a sequence of random variables independent and identically distributed of law
$$X_1(\mathbf{P}) = \frac{1}{4\pi(s^2 + 2rs)} \mathbf{1}_{D(2(r+s)) \setminus D(2r)}(x) dx,$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 4\pi(s^2 + 2rs)$ and independent of the sequence $\{X_n; n \geq 1\}$.

The arcs
$$\mathcal{A}_i = \frac{1}{2\pi(r+s)} \mathcal{A}(X_i), \quad i \geq 0,$$
are independent. The arc $\mathcal{A}_0$ is of constant length equal to

$$L_0 = l_{r,s} = \arccos(r/(r+s))/\pi.$$

The arcs $\mathcal{A}_i$, $i \geq 1$, are of length L_i, $i \geq 1$, having the distribution

$$\nu_{r,s}(dt) = \frac{\pi(r+s)^2}{2rs + s^2} \sin(2\pi t) \mathbf{1}_{[0, l_{r,s}]}(t) dt.$$

The corresponding probability of coverage is not contained in the framework of Siegel and Holst's formula. Nevertheless by adapting the proof of [23] to the case where one of the arcs has a constant length and the others have i.i.d. lengths, it is not too difficult to obtain the formula

$$
\begin{aligned}
&\mathbf{P}\left\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{0\le i\le n}\mathcal{A}_i\right\} \\
&= (1-a_{r,s})^n + \sum_{k=1}^{n}(-1)^k \binom{n}{k} \left\{ \int \mathbf{1}_{\{u_1 \ge l_{r,s}\}} \left[\prod_{i=2}^{k+1} F_{r,s}(u_i)\right] \left[\sum_{i=1}^{k+1}\int_0^{u_i} F_{r,s}(t)dt\right]^{n-k} d\sigma_{k+1}(u) \right. \\
&\qquad \left. - \int \left[\prod_{i=1}^{k} F_{r,s}(u_i)\right] \left[\sum_{i=1}^{k}\int_0^{u_i} F_{r,s}(t)dt\right]^{n-k} \left[\sum_{i=1}^{k}(u_i - l_{r,s})_+\right] d\sigma_k(u) \right\}, \quad (28)
\end{aligned}
$$

where

$$a_{r,s} = \mathbf{E}L_1 = \int t d\nu_{r,s}(t).$$

This last equality associated to (27) provides us (7).

□

Proof of Theorem 5. Fix $r, s > 0$. Remark that

$$L_i \le l_{r,s}, \quad \text{a.s. } i \ge 1.$$

Consequently, we obtain with Theorem 13 that

$$
\begin{aligned}
\mathbf{P}\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{i=0}^{n}\mathcal{A}_i\} &\le \mathbf{P}\{\exists x; ||x|| = 1 \text{ and } x \notin \cup_{i=1}^{n+1}\mathcal{A}_i\} \\
&= 1 - P(\nu_{r,s}, n+1) \\
&\le 1 - P(\delta_{a_{r,s}}, n+1), \quad n \ge 0, \qquad (29)
\end{aligned}
$$

where $a_{r,s} = \int t d\nu_{r,s}(t)$ denotes the expectation of $\nu_{r,s}$ given by the formula (10).

Besides, Shepp [21] proved by using a stopping-time argument that

Lemma 3 (Shepp, 1972) *If $0 \le a \le 1/4$, then we have*

$$1 - P(\delta_a, n) \le \frac{2(1-a)^{2n}}{\int_0^a (1-a-t)^n dt + (\frac{1}{4} - a)(1-2a)^n}, \quad n \ge 1.$$

With the choice $r/s \ge \cos(\pi/12)/(1-\cos(\pi/12))$, we have

$$a_{r,s} = \mathbf{E}L_1 \le l_{r,s} = \frac{1}{\pi}\arccos\left(\frac{r}{r+s}\right) \le \frac{1}{12},$$

so in particular

$$\frac{1}{4} - \frac{1}{n+1} - \left(1 - \frac{2}{n+1}\right)a_{r,s} \ge 0, \quad \forall n \ge 4.$$

Then by using Lemma 3, we obtain for all $n \ge 1$,

$$
\begin{aligned}
1 - P(\delta_{a_{r,s}}, n) &\le \frac{2(1-a_{r,s})^{2n}}{\int_0^{a_{r,s}}(1-a_{r,s}-t)^n dt + (\frac{1}{4} - a_{r,s})(1-2a_{r,s})^n} \\
&= \frac{2(1-a_{r,s})^{2n}}{\frac{1}{n+1}(1-a_{r,s})^{n+1} + (\frac{1}{4} - \frac{1}{n+1} - (1-\frac{2}{n+1})a_{r,s})(1-2a_{r,s})^n} \\
&\le \begin{cases} 2(n+1)(1-a_{r,s})^{n-1} & \text{for } n \ge 4 \\ 1 \le 2(n+1)(1-a_{r,s})^{n-1} & \text{for } n \le 3. \end{cases} \qquad (30)
\end{aligned}
$$

Consequently, the identity (27) and the inequalities (29) and (30) imply that

$$\mathbf{P}\{R_M \geq r+s|R_m=r\} \leq (8\pi(s^2+2rs)+4)e^{-4\pi(s^2+2rs)a_{r,s}},$$

and with the choice $s=1/r^\alpha$, $r^{1+\alpha} \geq \cos(\pi/12)/(1-\cos(\pi/12))$,

$$\mathbf{P}\{R_M \geq r+\frac{1}{r^\alpha}|R_m=r\} \leq (8\pi(r^{-2\alpha}+2r^{1-\alpha})+4)e^{-4\pi(r^{-2\alpha}+2r^{1-\alpha})a_{r,1/r^\alpha}}. \tag{31}$$

It remains to study the behaviour of $a_{r,1/r^\alpha}$, where $-1<\alpha<1/3$ is fixed, and r goes to infinity. We have

$$\begin{aligned} a_{r,1/r^\alpha} &= \frac{1}{2\sqrt{2}\pi}r^{\frac{1+\alpha}{2}}(1+\frac{1}{2r^{1+\alpha}})^{-1/2} \\ &\qquad +\frac{1}{\pi}\arccos(1-\frac{1}{1+r^{1+\alpha}})(1+\frac{1}{2r^{1+\alpha}})^{-1}(-\frac{r^{1+\alpha}}{4}+\frac{1}{2}+\frac{1}{4r^{1+\alpha}}) \\ &\sim \frac{1}{3\sqrt{2}\pi}\frac{1}{r^{\frac{1+\alpha}{2}}} \quad \text{when } r\to+\infty. \end{aligned}$$

This asymptotic result associated to (31) implies that for all $0<c<8/(3\sqrt{2})$ and all $-1<\alpha<1/3$,

$$\mathbf{P}\{R_M \geq r+\frac{1}{r^\alpha}|R_m=r\} = O(e^{-cr^{\frac{1}{2}(1-3\alpha)}}),$$

which provides the result of Theorem 5.

□

5 The case of the Crofton cell of a Poisson line process.

We are going to adapt our method to the study of the smallest disk centered at the origin and containing the Crofton cell of a Poisson line process in the plane. Let us recall first the required definitions.

Let Ψ a Poisson point process in $\mathbb{R}^2$, of intensity measure

$$\mu(A)=\int_0^{+\infty}\int_0^{2\pi}\mathbf{1}_A(\rho,\theta)d\theta d\rho, \quad A\in\mathcal{B}(\mathbb{R}^2). \tag{32}$$

For all $x\in\mathbb{R}^2$, let us consider

$$H(x)=\{y\in\mathbb{R}^2;(y-x)\cdot x=0\},$$

the polar line associated to x ($x\cdot y$ being the usual scalar product). Then the set $\mathcal{H}=\{H(x);x\in\Phi\}$ divides the space into convex polyhedra that constitute the so-called *two-dimensional Poissonian tessellation.*

In particular, this tessellation is isotropic, that means it is invariant by isometric transformations of $\mathbb{R}^2$. The first results on this geometrical object date from the beginning of the forties. There are due to Goudsmit [8] and to Miles [13], [14] and [15]. Some recent contributions can be found in [2], [6], [10], and [19].

Let us denote by C_0 the cell of the tessellation containing the origin. It can be proved [5] that the cell C_0 (known as *Crofton cell*) is almost surely well defined. Denote respectively by R'_m and R'_M the radii of the largest disk centered at the origin included in C_0 and the smallest disk centered at the origin containing C_0.

Proofs of Theorems 6 and 7. We use the same method of proof as for Theorems 1 and 2. The definition of C_0 provides us the following identity.

$$\{R'_M \geq r\} = \{\exists x; ||x|| = r/2 \text{ and } ||x-y|| \geq r/2 \;\; \forall y \in \Psi \cap D(r)\}. \tag{33}$$

Moreover, classically

$$\Psi \cap D(r) = \{X_n; 1 \leq n \leq N\}, \tag{34}$$

where:

(i) $\{X_n; n \geq 1\}$ is a sequence of i.i.d. random variables taking their values in $D(r)$ of law:

$$X_1(\mathbf{P}) = \frac{1}{2\pi r}\mathbf{1}_{D(r)}(x)d\mu(x);$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 2\pi r$ and independent of the sequence $\{X_n; n \geq 1\}$.

Then by using (33) and (34), we obtain that

$$\mathbf{P}\{R'_M \geq r\} = \mathbf{P}\{N = 0\} + \sum_{n\geq 1} \mathbf{P}\{N = n\} \times \mathbf{P}\{\exists x; ||x|| = r/2 \text{ and } x \notin \cup_{i=1}^n \mathcal{A}(X_i)\}.$$

It can be easily verified that the arcs $\mathcal{A}(X_i)$ are i.i.d. of respective lengths $\pi r L_i$ where

$$L_i = \frac{1}{\pi}\arccos\left(\frac{||X_i||}{r}\right), \quad i \geq 1,$$

has the distribution

$$\nu'_0(dt) = \pi \sin(\pi t)\mathbf{1}_{[0,1/2]}(t)dt.$$

We conclude as in the proof of Theorems 1 and 2.

□

Proof of Theorem 8. It consists in noticing that the law associated to the distribution function G is less concentrated about the mean than $\delta_{1/\pi}$ and more concentrated than

$$(1-\cos 1)\delta_0 + \cos 1\delta_{1/(\pi\cos 1)}.$$

Besides, by applying Stevens's formula (16) and its corollary (17), we obtain

$$1 - P(\delta_{1/\pi}, n) \;=\; n\left(1-\frac{1}{\pi}\right)^{n-1} - \binom{n}{2}\left(1-\frac{2}{\pi}\right)^{n-1} + \binom{n}{3}\left(1-\frac{3}{\pi}\right)^{n-1},$$

and on the other hand,

$$1 - P((1-\cos 1)\delta_0 + \cos 1\delta_{1/(\pi\cos 1)}, n) \;=\; (1-\cos 1)^n + n\cos 1\left(1-\frac{1}{\pi}\right)^{n-1}.$$

Then it remains to use Theorems 7 and 13.

$\square$

Proof of Theorem 9. The proof is based on the same arguments as for Theorem 4. We remark first the following identity

$$\{\Psi | R'_m = r\} \stackrel{law}{=} \Psi_r \cup \{X_0\},$$

where

(i) Ψ_r is a Poisson planar point process of intensity measure $\mathbf{1}_{D(r)^c} d\mu$

(ii) X_0 is a random variable uniformly distributed on the circle centered at the origin of radius r and independent of Φ_r.

So we see that the procedure described in the proof of Theorem 4 can be applied. We obtain that

$$\begin{aligned}
&\mathbf{P}\{R'_M \geq r+s | R'_m = r\} \\
&\qquad = \mathbf{P}\{\exists x; ||x|| = r+s \text{ and } x \notin \cup_{y \in \Psi_r \cup \{X_0\}} \mathcal{A}(y)\} \\
&\qquad \sum_{n \geq 0} \mathbf{P}\{N = n\} \times \mathbf{P}\{\exists x; ||x|| = r+s \text{ and } x \notin \cup_{0 \leq i \leq n} \mathcal{A}(X_i)\},
\end{aligned}$$

where

(i) $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables of law

$$X_1(\mathbf{P}) = \frac{1}{2\pi s} \mathbf{1}_{D(r+s) \setminus D(r)}(x) d\mu(x),$$

(ii) N is a Poisson variable of mean $\mathbf{E}N = 2\pi s$ and independent of the sequence $\{X_n; n \geq 1\}$.

The arcs

$$\mathcal{A}_i = \frac{1}{\pi(r+s)} \mathcal{A}(X_i), \quad i \geq 0,$$

are independent. The arc $\mathcal{A}_0$ is of constant length equal to

$$L_0 = l_{r,s} = \arccos(r/(r+s))/\pi.$$

The arcs $\mathcal{A}_i$, $i \geq 1$, are of lengths L_i, $i \geq 1$, of law

$$\nu'_{r,s}(dt) = \frac{\pi(r+s)}{s} \sin(\pi t) \mathbf{1}_{[0, l_{r,s}]}(t) dt.$$

We conclude then as for Theorem 4 by using (28).

$\square$

Proof of Theorem 10. As for Theorem 5, we notice first that

$$\mathbf{P}\{\exists x; ||x|| = r/2 \text{ and } x \notin \cup_{i=0}^{n} \mathcal{A}(X_i)\} \leq 1 - P(\delta_{b_{r,s}}, n+1), \quad n \geq 0.$$

Besides, using Lemma 3 for $r/s \geq \cos(\pi/12)/(1-\cos(\pi/12))$, we obtain that

$$1 - P(\delta_{b_{r,s}}, n) \leq 2(n+1)(1-b_{r,s})^{n-1}, \quad n \geq 1,$$

so

$$\mathbf{P}\{R'_M \geq r + s | R'_m = r\} \leq (4\pi s + 4)e^{-2\pi s b_{r,s}},$$

and with the choice $s = r^{\alpha}$, for $r^{1-\alpha} \geq \cos(\pi/12)/(1-\cos(\pi/12))$,

$$\mathbf{P}\{R'_M \geq r + r^{\alpha} | R'_m = r\} \leq (4\pi r^{\alpha} + 4)e^{-2\pi r^{\alpha} b_{r,s}}. \tag{35}$$

It remains to study the behaviour of $b_{r,r^{\alpha}}$, $\alpha \in (1/3, 1)$, when r goes to infinity. We have

$$\begin{aligned} b_{r,r^{\alpha}} &= \frac{\sqrt{2} r^{\frac{1-\alpha}{2}}}{\pi}(1 + \frac{1}{2r^{1-\alpha}})^{1/2} - \frac{r^{1-\alpha}}{\pi} \arccos(1 - \frac{1}{1+r^{1-\alpha}}) \\ &\sim \frac{4}{3\sqrt{2}\pi} \frac{1}{r^{\frac{1-\alpha}{2}}} \quad \text{when } r \to +\infty. \end{aligned} \tag{36}$$

Consequently, we obtain, considering (35) and (36) that for all $0 \leq c < 8/(3\sqrt{2})$,

$$\mathbf{P}\{R'_M \geq r + r^{\alpha} | R'_m = r\} = O(e^{-cr^{\frac{1}{2}(3\alpha-1)}}).$$

$\square$

References

[1] F. Baccelli and B. Blaszczyszyn. On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Rapport de recherche INRIA No 4019*, October 2000.

[2] P. Calka. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6):557–562, 2001.

[3] S. G. Foss and S. A. Zuyev. On a Voronoi aggregative process related to a bivariate Poisson process. *Adv. in Appl. Probab.*, 28(4):965–981, 1996.

[4] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33:958–972, 1962.

[5] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[6] A. Goldman. Sur une conjecture de D. G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne. *Ann. Probab.*, 26(4):1727–1750, 1998.

[7] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[8] S. Goudsmit. Random distribution of lines in a plane. *Rev. Modern Phys.*, 17:321–322, 1945.

[9] F. W. Huffer and L. A. Shepp. On the probability of covering the circle by random arcs. *J. Appl. Probab.*, 24(2):422–429, 1987.

[10] I. N. Kovalenko. A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons. *J. Appl. Math. Stochastic Anal.*, 12(4):301–310, 1999.

[11] S. Kumar and R. N. Singh. Thermal conductivity of polycristalline materials. *J. of the Amer. Cer. Soc.*, 78(3):728–736, 1995.

[12] J. L. Meijering. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.*, 8, 1953.

[13] R. E. Miles. Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci. U.S.A.*, 52:901–907, 1964.

[14] R. E. Miles. Random polygons determined by random lines in a plane. II. *Proc. Nat. Acad. Sci. U.S.A.*, 52:1157–1160, 1964.

[15] R. E. Miles. The various aggregates of random polygons determined by random lines in a plane. *Advances in Math.*, 10:256–290, 1973.

[16] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[17] A. Müller and D. Stoyan. *Comparison methods for stochastic models and risks.* John Wiley & Sons Ltd., Chichester, 2002.

[18] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams.* John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[19] K. Paroux. Quelques théorèmes centraux limites pour les processus Poissoniens de droites dans le plan. *Adv. in Appl. Probab.*, 30(3):640–656, 1998.

[20] E. Pielou. *Mathematical ecology.* Wiley-Interscience, New-York, 1977.

[21] L. A. Shepp. Covering the circle with random arcs. *Israel J. Math.*, 11:328–345, 1972.

[22] A. F. Siegel. Random space filling and moments of coverage in geometrical probability. *J. Appl. Probab.*, 15(2):340–355, 1978.

[23] A. F. Siegel and L. Holst. Covering the circle with random arcs of random sizes. *J. Appl. Probab.*, 19(2):373–381, 1982.

[24] H. Solomon. *Geometric probability.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1978. Ten lectures given at the University of Nevada, Las Vegas, Nev., June 9–13, 1975, Conference Board of the Mathematical Sciences—Regional Conference Series in Applied Mathematics, No. 28.

[25] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications.* John Wiley & Sons Ltd., Chichester, 1987. With a foreword by D. G. Kendall.

[26] R. van de Weygaert. Fragmenting the Universe III. The construction and statistics of 3-D Voronoi tessellations. *Astron. Astrophys.*, 283:361–406, 1994.

Chapitre 3

La fonction spectrale des mosaïques de Poisson-Voronoi et de Johnson-Mehl.

3.1 Présentation des résultats.

Les résultats présntés dans cette partie ont été obtenus en commun avec A. Goldman.

En 1996, A. Goldman a étudié le spectre du Laplacien avec condition de Dirichlet sur la cellule typique d'une mosaïque poissonienne de droites dans le plan [26]. Plus précisément, il a établi une formule exprimant l'espérance de la fonction spectrale $\varphi_0(t)$, $t > 0$, de cette cellule en fonction de la transformée de Laplace du périmètre de l'enveloppe convexe $\widehat{\mathbf{W}}$ de la trajectoire $\mathbf{W}$ d'un pont brownien entre 0 et 1, indépendant de la mosaïque :

$$\mathbf{E}\varphi_0(t) = \frac{1}{4\pi^2 t}\overline{\mathbf{E}}\exp\{-\sqrt{2t}V_1(\widehat{\mathbf{W}})\}, \tag{3.1}$$

où $\overline{\mathbf{E}}$ désigne l'espérance associée au pont brownien et $V_1(\widehat{\mathbf{W}})$ le périmètre de $\widehat{\mathbf{W}}$.

Tout repose sur la relation entre la fonction spectrale $\varphi_D(t)$, $t > 0$, d'un domaine convexe borné $D \subset \mathbb{R}^d$ et $\widehat{\mathbf{W}}$ [87] :

$$\varphi_D(t) = \frac{1}{(4\pi t)^{d/2}}\int_D \overline{\mathbf{P}}\{x + \sqrt{2t}\widehat{\mathbf{W}} \subset D\}dx, \tag{3.2}$$

où $\overline{\mathbf{P}}$ désigne la probabilité associée au pont brownien.

A. Goldman a déduit de l'égalité (3.1) un développement en série de $\mathbf{E}\varphi_0(t)$ pour tout $t > 0$, de même qu'un développement asymptotique au voisinage de zéro. Par ailleurs, il a mis en évidence une correspondance entre certaines propriétés de la mosaïque poissonienne de droites dans le plan d'une part - caractère circulaire de la cellule typique lorsque son volume est "grand", estimée asymptotique de la transformée de Laplace de sa première valeur propre- et l'enveloppe convexe de la trajectoire du pont brownien entre 0 et 1 d'autre part - caractère circulaire de $\widehat{\mathbf{W}}$ lorsque la trajectoire est "petite", estimée asymptotique de la transformée de Laplace du périmètre de $\widehat{\mathbf{W}}$ (voir [27], [28] pour plus de détails).

L'objectif de notre travail dans ce chapitre est de généraliser ces résultats aux mosaïques de Poisson-Voronoi d-dimensionnelles, $d \geq 2$, ainsi que, dans un certaine mesure, aux mosaïques de Johnson-Mehl. Dans l'article qui suit, on s'intéresse uniquement aux cellules de Voronoi. L'article placé en annexe (voir paragraphe 7.2) propose certaines généralisations au modèle de Johnson-Mehl.

Désignons par $\varphi(t)$, $t > 0$, la fonction spectrale de la cellule typique (aléatoire) $\mathcal{C}$ d'une mosaïque de Poisson-Voronoi dans $\mathbb{R}^d$, c'est-à-dire

$$\varphi(t) = \sum_{n \geq 0} e^{-t\lambda_n},$$

où $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ sont les valeurs propres successives du Laplacien avec condition au bord de Dirichlet sur $\mathcal{C}$. En appliquant la relation (3.2) à $\mathcal{C}$, on obtient pour toute dimension $d \geq 2$,

$$\mathbf{E}\varphi(t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int \overline{\mathbf{E}} \exp(-(2t)^{\frac{d}{2}} V_d(\widehat{\mathbf{W}}, x))dx, \quad t > 0, \tag{3.3}$$

où $V_d(C, x)$ désigne pour tout domaine convexe borné C de $\mathbb{R}^d$ la mesure de Lebesgue d-dimensionnelle de l'ensemble

$$\cup_{y \in C} B(y, ||y - x||).$$

Afin d'estimer plus précisément l'espérance dans l'égalité (3.3), il est nécessaire de calculer exactement le volume $V_d(C, x)$, $x \in \mathbb{R}^d$ pour tout convexe borné C. Appliquant ces formules au convexe particulier (aléatoire) $\widehat{\mathbf{W}}$, nous déduisons un développement asymptotique explicite à trois termes de $\mathbf{E}\varphi(t)$ au voisinage de zéro. En particulier, dans le cas $d = 2$, on vérifie que

$$\mathbf{E}\varphi(t) = \frac{\mathbf{E}V_2(\mathcal{C})}{4\pi t} - \frac{\mathbf{E}V_1(\mathcal{C})}{4\sqrt{4\pi t}} + \frac{1}{24}(\pi \mathbf{E}\alpha^{-1}(\mathcal{C}) - \mathbf{E}N_0(\mathcal{C}) + 2) + O(t^{-1/2}), \tag{3.4}$$

où $\alpha^{-1}(\mathcal{C})$ désigne la moyenne harmonique des angles de la cellule $\mathcal{C}$. En d'autres termes, le résultat (3.4) exprime le fait qu'il suffit de prendre l'espérance de chacun des termes du développement en zéro de la fonction spectrale déterministe pour retrouver celui de $\mathbf{E}\varphi(t)$.

Par ailleurs, toujours dans le cas deux-dimensionnel, la formule (3.3) permet d'obtenir un équivalent logarithmique de la transformée de Laplace de la première valeur propre λ_1, ainsi que de sa fonction de répartition :

$$\lim_{t \longrightarrow \infty} t^{-1/2} \ln \mathbf{E}\varphi(t) = \lim_{t \longrightarrow \infty} t^{-1/2} \ln \mathbf{E}e^{-t\lambda_1} = -4\sqrt{\pi} j_0, \tag{3.5}$$

et

$$\lim_{t \to 0} t \ln \mathbf{P}\{\lambda_1 \leq t\} = -4\pi {j_0}^2, \tag{3.6}$$

où j_0 est le premier zéro positif de la fonction de Bessel J_0. La preuve de ces deux derniers résultats repose d'une part sur une estimation de la fonction de répartition du périmètre de $\widehat{\mathbf{W}}$ dûe à A. Goldman [28] et d'autre part sur une application d'un lemme de T. W.

Anderson [1]. Ce lemme affirme que la probabilité pour un vecteur gaussien de covariance fixée d'appartenir à un ensemble convexe symétrique est maximale lorsque la moyenne du vecteur est nulle.

Les estimées (3.5) et (3.6) sont identiques pour la première valeur propre λ_1 de la cellule typique $\mathcal{C}$ et pour la première valeur propre μ_1 de la plus grande boule centrée à l'origine contenue dans $C(0)$ (égale en loi à $\mathcal{C}$). Ainsi, cela permet en un certain sens de confirmer que la cellule typique, lorsqu'elle a un volume "grand", est de forme approximativement circulaire. Un phénomène analogue, connu sous le nom de *conjecture de D. G. Kendall* [86], se produit pour une mosaïque poissonienne de droites dans le plan.

Enfin, certains des résultats précédemment énoncés dans le cas des mosaïques de Poisson-Voronoi peuvent être généralisés aux mosaïques de Johnson-Mehl. En particulier, en désignant toujours par $\varphi(t)$, $t > 0$, la fonction spectrale de la cellule typique et en notant Λ la mesure d'intensité temporelle de la mosaïque, on a

$$\mathbf{E}\varphi(t) = \frac{1}{\lambda(4\pi t)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \exp\left(-\int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right) \right\} dx\Lambda(du),$$

où $V_d(\sqrt{2t}\mathbf{W}, x, s)$, $x \in \mathbb{R}^d$, $s \geq 0$, est la mesure de Lebesgue d-dimensionnelle de l'ensemble

$$\bigcup_{\substack{y \in \mathbf{W} \\ ||y-x||+s>0}} B(y, ||y-x|| + s).$$

On en déduit un développement à deux termes de $\mathbf{E}\varphi(t)$ au voisinage de l'origine sous certaines hypothèses précises d'intégration portant sur la mesure Λ.

On the spectral function of the Poisson-Voronoi cells. *

André Goldman and Pierre Calka†

Abstract

Denote by $\varphi(t) = \sum_{n\geq 1} e^{-\lambda_n t}$, $t > 0$, the spectral function related to the Dirichlet laplacian for the typical cell $\mathcal{C}$ of a standard Poisson-Voronoi tessellation in $\mathbb{R}^d$, $d \geq 2$. We show that the expectation $\mathbf{E}\varphi(t)$, $t > 0$, is a functional of the convex hull of a standard d-dimensional Brownian bridge. This enables us to study the asymptotic behaviour of $\mathbf{E}\varphi(t)$, when $t \to 0^+, +\infty$. In particular, we prove that in the two-dimensional case ($d = 2$) the law of the first eigenvalue λ_1 of $\mathcal{C}$ satisfies the asymptotic relation $\ln \mathbf{E}e^{-t\lambda_1} \sim -t^{1/2} 4\sqrt{\pi} j_0$, when $t \to +\infty$, where j_0 is the first zero of the Bessel function J_0.

Introduction.

Consider $\Phi = \{x_n; n \geq 1\}$ a homogeneous Poisson point process in $\mathbb{R}^d$, $d \geq 2$, with the d-dimensional Lebesgue measure V_d for intensity measure. The set of cells

$$C(x) = \{y \in \mathbb{R}^d; ||y - x|| \leq ||y - x'||, x' \in \Phi\}, \quad x \in \Phi,$$

(which are almost surely bounded polyhedra) is the well-known *Poisson-Voronoi tessellation* of $\mathbb{R}^d$. Introduced by J. L. Meijering [15] and E. N. Gilbert [5] as a model of crystal aggregates, it provides now models for many natural phenomena as thermal conductivity [14], telecommunications [2], astrophysics [26] and ecology [21]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [24] and Okabe et al. [20].

In order to describe the statistical properties of the tessellation, the notion of *typical cell* $\mathcal{C}$ in the Palm sense is commonly used [17]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^d$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set

* *American Mathematical Society 2000 subject classifications.* Primary 35P15, 60D05, 60J65; secondary 52A22, 60G55.

Key words and phrases. Brownian bridge, convex hull, eigenvalues, empirical distributions, Johnson-Mehl tessellations, Laplace operator, Palm distribution, spectral function, stochastic geometry, Voronoi tessellations.

† *Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Andre.Goldman@univ-lyon1.fr, Pierre.Calka@univ-lyon1.fr

$B \subset \mathbb{R}^d$ such that $0 < V_d(B) < +\infty$. The typical cell $\mathcal{C}$ is defined by means of the identity [17]:

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{V_d(B)} \mathbf{E} \sum_{x \in B \cap \Phi} h(C(x) - x),$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

Consider now the cell

$$C(0) = \{y \in \mathbb{R}^d; ||y|| \leq ||y - x||, x \in \Phi\}$$

obtained when the origin is added to the point process Φ. It is well known [17] that $C(0)$ and $\mathcal{C}$ are equal in law. On the other hand, the typical cell can also be characterized by means of the empirical distributions. Indeed, let $\mathcal{V}_{d,R}$ be the set of all cells $C(x)$, $x \in \Phi$, included in the ball $B(R)$ centered at the origin and of radius $R > 0$. Let us define $N_R = \#\mathcal{V}_{d,R}$ and fix $h : \mathcal{K} \longrightarrow \mathbb{R}$ an arbitrary bounded measurable function. Then (see [8]):

$$\mathbf{E}h(\mathcal{C}) = \lim_{R \to +\infty} \frac{1}{N_R} \sum_{C(x) \in \mathcal{V}_{d,R}} h(C(x) - x), \quad \text{a.s..}$$

Despite of its popularity, the distributional statistical properties of the Poisson-Voronoi tessellation are mainly unknown and therefore have been intensively investigated by computing simulation [13], [18].

In the present work we study the properties of the *fundamental frequencies* of the typical cell $\mathcal{C}$. More precisely let us consider $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ...$ the (random) eigenvalues of the Laplacian under Dirichlet boundary conditions on $\mathcal{C}$ and denote by

$$\varphi(t) = \sum_{n \geq 1} e^{-\lambda_n t}, \quad t > 0,$$

the associated *spectral function*. On the other hand let

$$W(t) \in \mathbb{R}^d, \quad t \in [0, 1], W(0) \equiv W(1) \equiv 0,$$

be a d-dimensional Brownian bridge on the interval $[0, 1]$ independent of the point process Φ. Denote by

$$\mathbf{W} = \{W(t); 0 \leq t \leq 1\} \subset \mathbb{R}^d \tag{1}$$

the associated path and by $\widehat{\mathbf{W}}$ its closed convex hull. We show that the expectation $\mathbf{E}\varphi(t)$, $t > 0$, is a functional of the Lebesgue measure of the sets

$$B(\widehat{\mathbf{W}}; x) = \cup_{y \in \widehat{\mathbf{W}}} B(y, ||y - x||), \quad x \in \mathbb{R}^d,$$

where $B(y, R)$, $R > 0$ is the closed ball of $\mathbb{R}^d$, centered at $y \in \mathbb{R}^d$ and of radius $R > 0$. This enables us to study the asymptotic behaviour of $\mathbf{E}\varphi(t)$, when $t \to 0^+$ and when $t \to +\infty$ by using the properties of the closed convex hull $\widehat{\mathbf{W}}$.

In what follows, we denote by $\mathbf{P}$ (respectively $\overline{\mathbf{P}}$) the probability associated with the point process Φ (respectively with the Brownian bridge W). Likewise the expectations $\mathbf{E}$ and $\overline{\mathbf{E}}$ will refer respectively to Φ and W. For a set $D \subset \mathbb{R}^d$ we define

$$B(D; x) = \cup_{y \in D} B(y, ||y - x||), \quad \widetilde{B}(D; x) = \cup_{y \in D} \widetilde{B}(y, ||y - x||),$$

where $\widetilde{B}(y,R) = \{x \in \mathbb{R}^d; ||x-y|| < R\}$ is the open ball centered at y and of radius $R > 0$. To simplify notations, $V_d(D,x)$ will denote the d-dimensional Lebesgue measure of the set $\widetilde{B}(D;x)$, i.e.

$$V_d(D,x) = V_d(\widetilde{B}(D;x)).$$

The values of the expectations of the principal geometrical characteristics of the typical cell $\mathcal{C}$ are known [17]. In particular,

$$\mathbf{E}V_d(\mathcal{C}) = 1,$$

and denoting by $V_{d-1}(\mathcal{C})$ the $(d-1)$-dimensional area of the boundary $\partial\mathcal{C}$ and by $N_0(\mathcal{C})$ the number of vertices of $\mathcal{C}$ we have

$$\mathbf{E}V_{d-1}(\mathcal{C}) = \frac{\sqrt{\pi}d!\Gamma(2-1/d)\Gamma(d/2+1)^{-1/d}\Gamma(d/2)}{\Gamma((d+1)/2)\Gamma(d-1/2)}, \tag{2}$$

$$\mathbf{E}N_0(\mathcal{C}) = \frac{2^{d+1}\pi^{(d-1)/2}\Gamma((d^2+1)/2)\Gamma(d/2+1)^d}{d^2\Gamma(d^2/2)\Gamma((d+1)/2)^d}. \tag{3}$$

In particular, for a two-dimensional tessellation, $\mathbf{E}N_0(\mathcal{C}) = 6$.

Finally let us recall the value

$$\sigma_d = \nu_d(\mathbb{S}^{d-1}) = d\omega_d = 2\pi^{d/2}/\Gamma(d/2),$$

of the $(d-1)$-dimensional measure ν_d of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d; ||x|| = 1\}$ where $\omega_d = V_d(B(1))$ denotes the d-dimensional Lebesgue measure of the unit ball of $\mathbb{R}^d$.

The principal results of this paper were announced in [9]. Theorem 1 provides the expression of the expectation of the spectral function of the typical cell $\mathbf{E}\varphi(t)$ as a function of the convex hull of the Brownian bridge $\widehat{\mathbf{W}}$.

Theorem 1 *For $d \geq 2$,*

$$\begin{aligned}\mathbf{E}\varphi(t) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int \overline{\mathbf{E}}\exp(-(2t)^{\frac{d}{2}}V_d(\widehat{\mathbf{W}},x))dx \\ &= \frac{1}{(4\pi t)^{\frac{d}{2}}}\int \overline{\mathbf{E}}\exp(-V_d(\sqrt{2t}\widehat{\mathbf{W}},x))dx, \quad t>0.\end{aligned} \tag{4}$$

In the two following theorems, we give the small t asymptotics of $\mathbf{E}\varphi(t)$, and a large deviation result (when t goes to infinity) for the principal eigenvalue of the typical cell λ_1.

Theorem 2 *In dimension $d \geq 2$, we have when $t \longrightarrow 0^+$,*

$$\mathbf{E}\varphi(t) = \frac{\mathbf{E}(V_d(\mathcal{C}))}{(4\pi t)^{\frac{d}{2}}} - \frac{\mathbf{E}V_{d-1}(\mathcal{C})}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{c_{d,2}}{(4\pi)^{d/2}t^{\frac{d}{2}-1}} + O\left(\frac{1}{t^{\frac{d-3}{2}}}\right),$$

where

$$c_{d,2} = \left(4^d k_d\sigma_d\frac{\Gamma(3-\frac{2}{d})}{d\omega_d^{3-\frac{2}{d}}} - I_{d,2}\sigma_d\frac{\Gamma(2-\frac{2}{d})}{d\omega_d^{2-\frac{2}{d}}}\right)$$

with

$$k_d = 4\sigma_{d-1}\sigma_{d-2} \int_{\substack{0\le\varphi\le\varphi' \\ \varphi+\varphi'\le\pi}} \int_{\theta=\varphi'-\varphi}^{\varphi+\varphi'} \frac{\sin^2\theta}{2} \left[\frac{\theta(2\pi-\theta)}{6(\pi-\theta)} + \frac{1}{\tan\theta}\right]$$
$$\left[\sin^2\varphi\sin^2\varphi' - (\cos\theta - \cos\varphi\cos\varphi')^2\right]^{\frac{d-4}{2}} (\cos\varphi\cos\varphi')^{d-1}\sin\varphi\sin\varphi' d\theta d\varphi d\varphi'.$$

In particular, for $d = 2$,

$$c_{2,2} = \frac{2\pi}{3}\left(\int_0^{2\pi} \frac{1-\cos t}{t} dt - 1\right). \tag{5}$$

Theorem 3 *Denote by μ_1 the first eigenvalue of the largest random disc centered at the origin and included in $C(0)$. We have*

$$\begin{aligned} \lim_{t\longrightarrow\infty} t^{-1/2}\ln \mathbf{E}\varphi(t) &= \lim_{t\longrightarrow\infty} t^{-1/2}\ln \mathbf{E}e^{-t\lambda_1} \\ &= \lim_{t\longrightarrow\infty} t^{-1/2}\ln \mathbf{E}e^{-t\mu_1} \\ &= -4\sqrt{\pi}j_0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \lim_{t\to 0} t\ln \mathbf{P}\{\lambda_1 \le t\} &= \lim_{t\to 0} t\ln \mathbf{P}\{\mu_1 \le t\} \\ &= -4\pi {j_0}^2, \end{aligned} \tag{7}$$

where j_0 is the first positive zero of the Bessel function J_0.

In this work, we first give some useful preliminaries on the convex hull of the Brownian bridge. We then prove in the second section Theorem 1 and some easy consequences. The third (resp. fourth) section is devoted to the proof of Theorem 2 (resp. Theorem 3), which provides the asymptotic behaviour of $\mathbf{E}\varphi(t)$ when $t \to 0^+$ (resp. when $t \to +\infty$). In the last section, we enumerate some concluding remarks about the main results of the paper.

1 Preliminaries on the convex hull of the d-dimensional Brownian bridge.

We begin with the following elementary facts:

Proposition 1 *Consider a bounded closed set $C \subset \mathbb{R}^d$ and denote by $\widehat{C}$ its closed convex hull. Furthermore let $D \subset C$ be a countable, dense subset of C. We have*

$$B(C;x) = B(\widehat{C};x) \tag{8}$$
$$V_d(D,x) = V_d(C,x) = V_d(\overline{B(D;x)}), \quad x \in \mathbb{R}^d. \tag{9}$$

Proof of (8). Let $u \in \mathbb{S}^{d-1}$ be a unit vector such that

$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) \neq \emptyset.$$

It can be see easily that there exist a point $z \in x + \mathbb{R}_+ u$ and a support hyperplane H_u of $\widehat{C}$ perpendicular to u such that:

(i)
$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) = \overline{xz},$$
where $\overline{xz} = \{\lambda x + (1-\lambda)z, 0 \leq \lambda \leq 1\}$ is the closed segment with bounding points x, z;

(ii)
$$||y - z|| = ||y - x||, \quad \forall y \in H_u \cap \partial\widehat{C}.$$

Moreover it is known ([23], corollary 18.3.1) that the intersection $H_u \cap \partial\widehat{C}$ must contain at least one point $y \in C$. Therefore

$$(x + \mathbb{R}_+ u) \cap B(\widehat{C}; x) \subset B(C; x) \quad \forall u \in \mathbb{S}^{d-1},$$

which implies (8).

Proof of (9). Fix $y \in \overline{B(D; x)}$. An elementary geometrical argument shows (the set D being bounded) that

$$\{\lambda x + (1-\lambda)y; 0 < \lambda \leq 1\} \subset B(D; x).$$

Integrating then in spherical coordinates (with x as center) we obtain the equality

$$V_d(D, x) = V_d(\overline{B(D; x)}).$$

Combining this with the obvious inclusions

$$\widetilde{B}(D; x) \subset \widetilde{B}(C; x) \subset B(C; x) \subset \overline{B(D; x)}, \quad x \in \mathbb{R}^d,$$

we obtain the result.

□

Our next task is to give a useful estimate of the difference

$$V_d(C, x) - \omega_d ||x||^d = V_d(B(C; x) \setminus B(||x||))$$

for sets $C \subset \mathbb{R}^d$ containing the origin. It follows from (8) that we may suppose that the set C is convex. Let us introduce a few notations. Fix $x \in \mathbb{R}^d \setminus \{0\}$ and define:

(i) $H = \left\{y \in \mathbb{R}^d; (y - x) \cdot x = 0\right\}$ the polar hyperplane of the point x;

(ii) $H^+ = \left\{y \in \mathbb{R}^d; (y - x) \cdot x \leq 0\right\}$ the half-space associated with H and containing the origin;

(iii) $\mathbb{S}^+ = \mathbb{S}^{d-1} \cap (H^+ - x), \quad \mathbb{S}^- = \mathbb{S}^{d-1} \setminus \mathbb{S}^+$;

(iv) $H_{0,u}$ the hyperplane perpendicular to the vector $u \in \mathbb{S}^{d-1}$ and containing the origin;

(v) H_u the support hyperplane of C perpendicular to u and included in the half-space $H^+_{0,u} = \{y \in \mathbb{R}^d; y \cdot u \geq 0\}$;

(vi) $\mathcal{A}(C, x) = \{u \in \mathbb{S}^-; H_u \cap (x + \mathbb{R}_+ u)\} \neq \emptyset\}$ (notice that $H_u \cap (x + \mathbb{R}_+ u) \neq \emptyset$ for all $u \in \mathbb{S}^+$);

(vii) $m(x, u) = d(x, H_u)$ the distance between x and H_u,
$\rho(x, u) = |x.u| = d(x, H_{0,u})$ the distance between x and $H_{0,u}$,
$h(u) = d(0, H_u)$ the distance between H_u and $H_{0,u}$.

Remark 1 For all $x \in \mathbb{R}^d \setminus \{0\}$, we have

$$\begin{cases} m(x,u) = & h(u) + \rho(x,u) & \textit{if } u \in \mathbb{S}^+, \\ m(x,u) = & |h(u) - \rho(x,u)| & \textit{if } u \in \mathbb{S}^-, \\ \mathcal{A}(C,x) = & \{u \in \mathbb{S}^-; h(u) - \rho(x,u) \geq 0\} & . \end{cases} \tag{10}$$

Proposition 2 *For $x \in \mathbb{R}^d$, we have:*

$$\begin{aligned} V_d(C, x) - \omega_d ||x||^d \quad = \quad & \frac{2^d}{d} \sum_{j=1}^{d} \binom{d}{j} \int_{\mathbb{S}^+} h(u)^j \rho(x,u)^{d-j} d\nu_d(u) \\ & + \frac{2^d}{d} \int_{\mathbb{S}^-} [(h(u) - \rho(x,u)) \vee 0]^d d\nu_d(u), \end{aligned} \tag{11}$$

and in particular,

$$V_d(C, 0) = \frac{2^d}{d} \int_{\mathbb{S}^{d-1}} h(u)^d d\nu_d(u). \tag{12}$$

Proof. For $u \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d \setminus \{0\}$, three possibilities occur:

Case 1 $u \notin \mathbb{S}^+ \cup \mathcal{A}(C, x)$ and consequently

$$(x + \mathbb{R}_+ u) \cap B(C; x) = (x + \mathbb{R}_+ u) \cap B(||x||) = \{x\}.$$

Case 2 $u \in \mathbb{S}^+$ which implies that

$$(x + \mathbb{R}_+ u) \cap B(C; x) = \overline{xz}, \quad (x + \mathbb{R}_+ u) \cap B(||x||) = \overline{xz'},$$

with

$$||x - z|| = 2m(x, u), \quad ||x - z'|| = 2\rho(x, u).$$

Case 3 $u \in \mathcal{A}(C, x)$ which implies that

$$(x + \mathbb{R}_+ u) \cap B(C; x) = \overline{xz}, \quad (x + \mathbb{R}_+ u) \cap B(||x||) = \{x\},$$

with

$$||x - z|| = 2m(x, u).$$

Then integration in spherical coordinates (with x as center) gives that

$$\begin{aligned}
V_d(C,x) &- \omega_d ||x||^d \\
&= \int_{\mathbb{S}^+} \left[\int_{2\rho(x,u)}^{2m(x,u)} r^{d-1} dr \right] d\nu_d(u) + \int_{\mathcal{A}(C,x)} \left[\int_0^{2m(x,u)} r^{d-1} dr \right] d\nu_d(u) \\
&= (2^d/d) \left[\int_{\mathbb{S}^+} (m(x,u)^d - \rho(x,u)^d) d\nu_d(u) + \int_{\mathcal{A}(C,x)} m(x,u)^d d\nu_d(u) \right]. \quad (13)
\end{aligned}$$

From (10) we get that

$$m(x,u)^d - \rho(x,u)^d = \sum_{j=1}^{d} \binom{d}{j} h(u)^j \rho(x,u)^{d-j}, \quad u \in \mathbb{S}^+,$$

and

$$(h(u) - \rho(x,u)) \vee 0 = \begin{cases} m(x,u) & \text{for } u \in \mathcal{A}(C,x) \\ 0 & \text{for } u \in \mathbb{S}^- \setminus \mathcal{A}(C,x). \end{cases}$$

Substituting these expressions in (13) we find the final result (11).
To prove (12), it suffices to notice that for all $u \in \mathbb{S}^{d-1}$, we have

$$\mathbb{R}_+ u \cap B(C;0) = \overline{0z}$$

with

$$||z|| = 2h(u),$$

so integrating in spherical coordinates we obtain the result.

□

Let us now suppose that C is a random convex set containing the origin and invariant (in law) by rotations with the origin as center. Thus the random variables $h(u)$, $u \in \mathbb{S}^{d-1}$, are equal in law and we obtain:

Proposition 3 *Suppose that C satisfies the above conditions and that $\mathbf{E}\{h(u)^d\} < \infty$, $u \in \mathbb{S}^{d-1}$. Fixing $u_0 \in \mathbb{S}^{d-1}$, we then have*

$$\begin{aligned}
\mathbf{E}\left(V_d(C,x) - \omega_d ||x||^d\right) &= \sum_{j=1}^{d} I_{d,j} ||x||^{d-j} \mathbf{E}\left(h(u_0)^j\right) \\
&\quad + \frac{2^d}{d} \int_{\mathbb{S}_-} \mathbf{E}\left([(h(u_0) - \rho(x,u)) \vee 0]^d\right) d\nu_d(u)
\end{aligned}$$

where

$$I_{d,j} = \frac{2^d}{d} \sigma_{d-1} \binom{d}{j} \int_0^1 t^{d-j} (1-t^2)^{\frac{d-3}{2}} dt = \frac{2^d}{d} \binom{d}{j} \frac{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{d+1-j}{2}\right)}{\Gamma\left(d - \frac{j}{2}\right)}, \quad 1 \leq j \leq d.$$

Proof. Taking the expectation in (11) the result follows from the direct evaluation of the integrals

$$\int_{\mathbb{S}^+} \rho(x,u)^{d-j} d\nu_d(u) = ||x||^{d-j} I_{d,j}.$$

$\square$

Remark 2 Under the conditions stated in Proposition 3 we obtain that

$$\mathbf{E}\left(V_d(\varepsilon C, x) - \omega_d ||x||^d\right) \sim_{\varepsilon \longrightarrow 0} I_{d,1}\varepsilon ||x||^{d-1} \mathbf{E}h(u_0).$$

In particular, in dimension $d = 2$, it follows from the Cauchy formula giving the perimeter of a convex set that

$$\mathbf{E}\left(V_2(\varepsilon C, x) - \pi ||x||^2\right) \sim_{\varepsilon \longrightarrow 0} \varepsilon \frac{4||x||}{\pi} \mathbf{E}V_1(C),$$

where $V_1(C)$ denotes the perimeter of the convex set C.

Choose now for C the closed convex hull $\widehat{\mathbf{W}} \subset \mathbb{R}^d$, of the sample path of the d-dimensional Brownian bridge on the interval $[0,1]$. Recall that $\widehat{\mathbf{W}}$ is invariant by rotations with the origin as center. Hence the random variables $h(u)$, $u \in \mathbb{S}^{d-1}$, defined above coincide in law with the maximum M_0 of the 1-dimensional Brownian bridge. The law of M_0 is explicitely known, namely [25]

$$\overline{\mathbf{P}}\{M_0 \geq u\} = e^{-2u^2}. \tag{14}$$

Hence all the moments of M_0 are finite, and we have

$$\overline{\mathbf{E}}{M_0}^{2k} = \left(\frac{1}{2}\right)^k k!, \quad \overline{\mathbf{E}}{M_0}^{2k+1} = \frac{(2k+1)!}{8^k k!} \frac{\sqrt{\pi}}{2\sqrt{2}}, \quad k \in \mathbb{N}.$$

In particular,

$$\overline{\mathbf{E}}\left(V_d(\varepsilon \widehat{\mathbf{W}}, x) - \omega_d ||x||^d\right) \sim_{\varepsilon \longrightarrow 0} I_{d,1}\varepsilon ||x||^{d-1} \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Now, denote by

$$M = \sup_{y \in \widehat{\mathbf{W}}} ||y|| = \sup_{0 \leq s \leq 1} ||W(s)||$$

the maximum of the radial part of the Brownian bridge

$$W(s) = (W_1(s), \cdots, W_d(s)), \quad 0 \leq s \leq 1.$$

The components $W_i(s)$, $0 \leq s \leq 1$, $i = 1, \cdots, d$, are independent one-dimensional Brownian bridges. Hence

$$\begin{aligned} \overline{\mathbf{P}}\{M \geq s\} &\leq d\overline{\mathbf{P}}\{\sup_{0 \leq s \leq 1} |W_1(s)| \geq s/\sqrt{d}\} \\ &\leq 2d\overline{\mathbf{P}}\{M_0 \geq s/\sqrt{d}\} \\ &= 2de^{-2s^2/d}, \quad s \geq 0, \end{aligned} \tag{15}$$

and

$$\overline{\mathbf{E}}M^k < +\infty, \quad \forall k \geq 0. \tag{16}$$

As a consequence we deduce the following result:

Proposition 4 *For all $k \in \mathbb{N}^*$ there exists a constant $0 < c_k < +\infty$ such that*

$$\overline{\mathbf{E}}|V_d(\varepsilon\widehat{\mathbf{W}}, x) - \omega_d||x||^d|^k \leq c_k \sum_{i=k}^{kd} \varepsilon^i ||x||^{kd-i},$$

$\varepsilon > 0$, $x \in \mathbb{R}^d$.

Proof. From Proposition 2 we have

$$|V_d(\varepsilon\widehat{W}, x) - \omega_d||x||^d| \leq (2^d/d) \left[\sum_{j=1}^{d-1} \binom{d}{j} (\varepsilon M)^j \int_{\mathbb{S}^+} \rho(x,u)^{d-j} d\nu_d(u) + \omega_d(\varepsilon M)^d\right],$$

which by (16) implies the result.

$\square$

2 Proof of Theorem 1 and consequences.

Proof of Theorem 1. Consider the spectral function

$$\varphi(t) = \sum_{n \geq 1} e^{-t\lambda_n}, \quad t > 0,$$

of the typical cell $\mathcal{C}$.

Let us recall first (see [6], [25]) that the spectral function $\varphi_U(t)$, $t > 0$, of any bounded domain $U \subset \mathbb{R}^d$ can be expressed in term of the Brownian bridge under the form:

$$\varphi_U(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_U \overline{\mathbf{P}}\{x + \sqrt{2t}\mathbf{W} \subset U\} dx. \tag{17}$$

Applying the formula above to the domain

$$C(0) = \{y \in \mathbb{R}^d; ||y|| \leq ||y - x||, x \in \Phi\} \overset{law}{=} \mathcal{C},$$

and taking the expectation we obtain (by Fubini theorem)

$$\mathbf{E}\varphi(t) \quad = \quad \frac{1}{(4\pi t)^{d/2}} \overline{\mathbf{E}} \int \mathbf{P}\{x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} dx$$

Observe that

$$-x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0) \Longleftrightarrow \Phi \cap \{-x + B(\sqrt{2t}\widehat{\mathbf{W}}, x)\} = \emptyset.$$

Therefore applying the property of the Poisson point process Φ we obtain

$$\mathbf{P}\{-x + \sqrt{2t}\widehat{\mathbf{W}} \subset C(0)\} = \exp\{-V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)\},$$

and consequently

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{d/2}} \int \overline{\mathbf{E}} \exp\left\{-V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)\right\} dx.$$

The obvious identity

$$V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) = (2t)^{d/2} V_d(\widehat{\mathbf{W}}, x/\sqrt{2t}), \quad t > 0, x \in \mathbb{R}^d,$$

and an elementary change of variable provide the result.

□

In order to study the asymptotics of $\mathbf{E}\varphi(t)$, when $t \to 0^+$, we derive from (4) the following suitable relation.

Theorem 4 *For $k \geq 1$, the following asymptotic, when $t \to 0^+$, holds:*

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \left\{ \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \int \overline{\mathbf{E}}\{(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d)^i\} e^{-\omega_d ||x||^d} dx + O(t^{k/2}) \right\}.$$

Proof. Let us start with the formula

$$\begin{aligned} \mathbf{E}\varphi(t) &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int \overline{\mathbf{E}} \exp\left\{-(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x)\right\} dx \\ &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int \overline{\mathbf{E}} \exp\left\{-(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d)\right\} e^{-\omega_d ||x||^d} dx. \end{aligned} \quad (18)$$

Fix $k \geq 1$. Since

$$V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d \geq 0, \quad x \in \mathbb{R}^d,$$

we have

$$\begin{aligned} &|\exp\left\{-\left(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d\right)\right\} - \sum_{i=0}^{k-1} \frac{(-1)^i}{i!} \left(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d\right)^i | \\ &\qquad \leq \frac{1}{k!} \left(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d\right)^k. \end{aligned} \quad (19)$$

By Proposition 4 we have

$$\int \overline{\mathbf{E}} \left\{ \left[V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d \right]^i \right\} e^{-\omega_d ||x||^d} dx < +\infty, \quad \forall i \in \mathbb{N},$$

and

$$\int \overline{\mathbf{E}} \left\{ \left[V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d \right]^k \right\} e^{-\omega_d ||x||^d} dx \leq c'_k t^{k/2}, \quad 0 < t \leq 1/2, \quad (20)$$

where $0 < c'_k < +\infty$ is a constant. The result follows then from (18), (19) and (20).

□

In the two-dimensional case $d = 2$, the result expressed by Theorem 4 can be reinforced in the following form.

Theorem 5 *For $d = 2$ there exists $t_0 > 0$ such that:*

$$\mathbf{E}\varphi(t) = \frac{1}{4\pi t}\sum_{n\geq 0}\frac{(-1)^n}{n!}\int \overline{\mathbf{E}}(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - \pi||x||^2)^n e^{-\pi||x||^2}dx, \tag{21}$$

for all $0 < t < t_0$, the series being absolutely convergent.

Proof. Regarding (4) it suffices to prove that there exists $t_0 > 0$ such that

$$\int \overline{\mathbf{E}}\exp(V_2(\sqrt{2t_0}\widehat{\mathbf{W}}, x) - 2\pi||x||^2)dx < +\infty.$$

Using the obvious inclusion

$$B(\sqrt{2t}\widehat{\mathbf{W}}; x) \subset B(2\sqrt{2t}M + ||x||),$$

we obtain the inequality

$$V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) \leq \pi(||x|| + 2\sqrt{2t}M)^2 \leq \frac{3\pi}{2}||x||^2 + 24t\pi M^2. \tag{22}$$

Hence,

$$\begin{aligned}
\int \overline{\mathbf{E}}\exp(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - 2\pi||x||^2)dx &\leq \int \overline{\mathbf{E}}\exp(V_2(\sqrt{2t}\widehat{\mathbf{W}}, x) - \frac{\pi}{2}||x||^2)dx \\
&\leq \overline{\mathbf{E}}\exp(24t\pi M^2) \\
&= \int_0^{+\infty} e^s\overline{\mathbf{P}}\{24\pi M^2 \geq s/t\}ds + 1 \\
&\leq 1 + 4\int_0^{+\infty} e^s e^{-s/(24\pi t)}ds,
\end{aligned}$$

by using (15). Thus it suffices to take $t_0 < \frac{1}{24\pi}$.

□

In the two-dimensional case $d = 2$ the formula (4) provides straightforwardly an identification, in term of Brownian bridge, of the expectation of the distribution function of the eigenvalues

$$N(t) = \sum_{n\geq 1} 1_{\{\lambda_n \leq t\}}, t > 0,$$

Indeed on one hand we have

$$\mathbf{E}\varphi(t) = t\int_0^{+\infty} e^{-ts}\mathbf{E}\sum_{n\geq 1} 1_{\{\lambda_n\leq s\}}ds, \quad t > 0.$$

On the other hand an elementary computation yields

$$\frac{1}{2\pi}\int e^{-2tV_2(\widehat{\mathbf{W}},x)}dx = \frac{t}{2\pi}\int_0^{+\infty} e^{-ts}\left(\int \overline{\mathbf{P}}\{2V_2(\widehat{\mathbf{W}}, x) \leq s\}dx\right)ds.$$

So by injectivity of the Laplace transform and Theorem 1 we obtain

Theorem 6 *In dimension $d = 2$, the expectation of the distribution function $N(s), s > 0$, is of the form*

$$\mathbf{E}N(t) = \frac{1}{2\pi} \int \overline{\mathbf{P}}\{2V_2(\widehat{\mathbf{W}}, x) \leq t\} dx.$$

3 Proof of Theorem 2.

Recall that H. Weyl established in 1911 [28] that for a bounded domain $U \subset \mathbb{R}^d$ with piecewise smooth boundary the spectral function φ_U (of the Dirichlet Laplacian on U) satisfies the asymptotic relation

$$\varphi_U(t) \sim_{t \to 0^+} \frac{V_d(U)}{(4\pi t)^{d/2}}.$$

For a bounded polygonal convex domain in $\mathbb{R}^2$, it was shown (see [11], [22], [27]) that when $t \to 0^+$,

$$\varphi_U(t) = \frac{V_2(U)}{4\pi t} - \frac{V_1(U)}{4\sqrt{4\pi t}} + \frac{1}{24}(\pi \alpha^{-1}(U) - N_0(U) + 2) + O(e^{-c/t}), \tag{23}$$

where

(i) $V_1(U)$ is the perimeter of U;

(ii) $N_0(U)$ is the number of vertices of U;

(iii) $\alpha^{-1}(U) = \sum_{i=1}^{N_0(U)} 1/\alpha_i$ is the harmonic mean of the inside-facing angles $\alpha_1, \ldots, \alpha_{N_0(U)}$ at the vertices of U;

(iv) $c > 0$ is a positive constant independent of $t > 0$.

At last, B. U. Fedosov [4] proved that for a bounded convex non-degenerate polyhedron $U \subset \mathbb{R}^d$, $d \geq 3$, we have when $t \to 0^+$,

$$\varphi_U(t) = \frac{V_d(U)}{(4\pi t)^{\frac{d}{2}}} - \frac{V_{d-1}(U)}{4(4\pi t)^{\frac{d-1}{2}}} + \frac{1}{8(4\pi t)^{\frac{d-2}{2}}} \sum_{i=1}^{N_{d-2}(U)} \frac{1}{3}\left(\frac{\omega_i}{\pi} - \frac{\pi}{\omega_i}\right) V_{d-2}(F_i) + O\left(\frac{1}{t^{\frac{d-3}{2}}}\right), \tag{24}$$

where

(i') $V_{d-1}(U)$ is the $(d-1)$-dimensional measure of the boundary of U;

(ii') F_i, $i = 1, \cdots, N_{d-2}(U)$ are the $(d-2)$-dimensional faces of U;

(iii') ω_i is the magnitude of the dihedral angle at the face F_i, $1 \leq i \leq N_{d-2}(U)$.

Returning now to the spectral function of the typical cell $\mathcal{C}$, Theorem 4 and Proposition 3 tell us that for all $k \geq 1$ we have the asymptotic relation (when $t \to 0^+$)

$$\mathbf{E}\varphi(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \left\{ \sum_{i=0}^{k-1} c_{d,i} t^{i/2} + O(t^{k/2}) \right\}. \tag{25}$$

Meanwhile the explicit computation of the coefficients $c_{d,i}$ requires the values of the covariances

$$c(u_1, \cdots, u_j) = \overline{\mathbf{E}}(M_{u_1} \cdots M_{u_j}), \quad u_1, \cdots, u_j \in \mathbb{S}^{d-1}, 1 \leq j \leq i,$$

where

$$M_u = \sup_{0 \leq s \leq 1} (u \cdot W(s)), \quad u \in \mathbb{S}^{d-1},$$

denotes the projection of the d-dimensional Brownian path $\mathbf{W}$ on the half-line $\mathbb{R}_+ u$. To our knowledge, only $c(u, u')$, $u, u' \in \mathbb{S}^{d-1}$, is explicitely calculated (see [6]).

Proof of Theorem 2. Theorem 4 provides us the following expansion:

$$\begin{aligned}\mathbf{E}\varphi(t) \quad = \quad & \frac{1}{(4\pi t)^{\frac{d}{2}}} \left[1 - \int \overline{\mathbf{E}}(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d) e^{-\omega_d ||x||^d} dx \right. \\ & \left. + \frac{1}{2} \int \overline{\mathbf{E}}\{(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d)^2\} e^{-\omega_d ||x||^d} dx + O(t\sqrt{t}) \right] . \end{aligned} \tag{26}$$

Applying Proposition 3 to $C = \sqrt{2t}\widehat{\mathbf{W}}$ we obtain

$$\begin{aligned}\overline{\mathbf{E}}(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d) \quad = \quad & \sum_{j=1}^{d} I_{d,j} ||x||^{d-j} (2t)^{j/2} \overline{\mathbf{E}} {M_0}^j \\ & + \frac{2^d}{d} \int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t} M_u - \rho(x, u)) \vee 0]^d d\nu_d(u).\end{aligned}$$

Suppose $d \geq 3$. Since

$$\begin{aligned}&\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t} M_u - \rho(x, u)) \vee 0]^d d\nu_d(u) \leq (2t)^{d/2} \sigma_d \overline{\mathbf{E}} M^d, \\ &\overline{\mathbf{E}} M_0 = \frac{\sqrt{\pi}}{2\sqrt{2}} \quad \text{ and } \overline{\mathbf{E}} {M_0}^2 = \frac{1}{2},\end{aligned}$$

then there exist $t_0 > 0$ and a constant $K_d > 0$ such that the expression above is of the form

$$\overline{\mathbf{E}} \left(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d ||x||^d \right) = \frac{\sqrt{\pi t}}{2} ||x||^{d-1} I_{d,1} + t ||x||^{d-2} I_{d,2} + t\sqrt{t} A_d(x, t), \tag{27}$$

with

$$0 \leq A_d(x, t) \leq K_d \left(1 + ||x||^{d-3} \right), \quad x \in \mathbb{R}^d, \quad 0 < t < t_0. \tag{28}$$

For $d = 2$ let us note

$$K(t, ||x||) = \frac{\sqrt{2t} M_0}{||x||}, \quad t > 0, x \in \mathbb{R}^d \setminus \{0\}.$$

We have

$$
\begin{aligned}
&\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u)) \vee 0]^2 d\nu_2(u) \\
&= 2\mathbf{E}\left\{4tM_0{}^2 \arcsin\left(K(t,||x||)\wedge 1\right) + ||x||^2 \left[\arcsin\left(K(t,||x||)\wedge 1\right)\right.\right. \\
&\quad \left.\left. -K(t,||x||)\sqrt{(1-K(t,||x||)^2)\wedge 0}\right] - 4||x||\sqrt{2t}M_0 \left[1-\sqrt{(1-K(t,||x||)^2)\wedge 0}\right]\right\} \\
&\leq 2\overline{\mathbf{E}}\left\{4tM_0^2 \arcsin\left(K(t,||x||)\wedge 1\right)\right. \\
&\qquad\qquad \left. +||x||^2\left[\arcsin\left(K(t,||x||)\wedge 1\right) - K(t,||x||)\sqrt{(1-K(t,||x||)^2)\wedge 0}\right]\right\}.
\end{aligned}
$$

Considering the two cases $K(t,||x||) \geq (1/2)||x||$ and $K(t,||x||) < (1/2)||x||$, some elementary and somewhat lengthy calculations (using in particular the existence of a constant $\alpha > 0$ verifying

$$\arcsin(x) \leq x + \alpha x^3, \quad \forall x \in [0, \frac{1}{2}].$$

and the inequality $\sqrt{1-x^2} \geq (1-x^2)$, $0 \leq x \leq 1$) provide the following estimation

$$\int_{\mathbb{S}^-} \overline{\mathbf{E}}[(\sqrt{2t}M_u - \rho(x,u)) \vee 0]^2 d\nu_2(u) \leq C\frac{t\sqrt{t}}{||x||}. \tag{29}$$

where $C > 0$ is a constant.

The inequality (29) shows that the formulas (27) and (28) are valid alike for $d = 2$.

Now we may write (11) on the form

$$V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d||x||^d = 2^d\sqrt{2t}\int_{\mathbb{S}^+} M_u \rho(x,u)^{d-1} d\nu_d(u) + R_d(x,t)$$

with

$$0 \leq R_d(x,t) \leq \frac{2^d}{d}\sigma_d \sum_{j=2}^{d} \binom{d}{j} (\sqrt{2t}M)^j ||x||^{d-j} I_{d,j}.$$

Hence there exists $t_0 > 0$ and a constant $K'_d > 0$ such that

$$\overline{\mathbf{E}}\left\{(V_d(\sqrt{2t}\widehat{\mathbf{W}}, x) - \omega_d||x||^d)^2\right\} = 4^d k_d ||x||^{2d-2} 2t + t\sqrt{t}G_d(x,t), \quad x \in \mathbb{R}^d, \ t > 0, \tag{30}$$

with

$$k_d = \frac{1}{||x||^{2d-2}} \iint_{(\mathbb{S}^+)^2} \overline{\mathbf{E}}(M_u M_{u'})(\rho(x,u)\rho(x,u'))^{d-1} d\nu_d(u) d\nu_d(u'), \tag{31}$$

and

$$0 \leq G_d(x,t) \leq K'_d\left(1 + ||x||^{2d-3}\right), \quad x \in \mathbb{R}^d, \quad 0 < t < t_0.$$

The covariances $\overline{\mathbf{E}}(M_u M_{u'})$ were calculated in ([6], IV.). Precisely, if $\theta \in (0,\pi]$ is the angle spanned by the two vectors $u, u' \in \mathbb{S}^{d-1}$, then

$$\overline{\mathbf{E}}(M_u M_{u'}) = H(\theta) = \frac{\sin\theta}{2}\left[\frac{\theta(2\pi-\theta)}{6(\pi-\theta)} + \frac{1}{\tan\theta}\right]. \tag{32}$$

Inserting (32) in the integral (31) some calculation yields:

$$k_d = 4\sigma_{d-1}\sigma_{d-2} \int_{\substack{0\le\varphi\le\varphi' \\ \varphi+\varphi'\le\pi}} \int_{\theta=\varphi'-\varphi}^{\varphi+\varphi'} H(\theta)\sin\theta \left[\sin^2\varphi\sin^2\varphi' - (\cos\theta-\cos\varphi\cos\varphi')^2\right]^{\frac{d-4}{2}} d\theta$$

$$(\cos\varphi\cos\varphi')^{d-1}\sin\varphi\sin\varphi' d\varphi d\varphi', \quad d\ge 3, \tag{33}$$

$$k_2 = \frac{\pi^2}{12}\left(1+\frac{1}{2}\int_0^{2\pi}\frac{1-\cos u}{u}du\right). \tag{34}$$

In order to obtain Theorem 2 it suffices now to insert formulas (27) and (30) in (26), and to proceed to some elementary calculations thanks to (33) and (34).

□

Remark 3 The asymptotic result of Theorem 2 and the equation (23) suggest that we may have

$$c_{2,2} = \frac{4\pi}{24}(\pi\mathbf{E}\alpha^{-1}(\mathcal{C}) - \mathbf{E}N_0(\mathcal{C}) + 2).$$

By (3) and (34) this is equivalent to the equality

$$\mathbf{E}\alpha^{-1}(\mathcal{C}) = \frac{4}{\pi}\int_0^{2\pi}\frac{1-\cos t}{t}dt. \tag{35}$$

We did not find this result in the literature so for the sake of completeness we give its proof in the Appendix.

4 Proof of Theorem 3.

The main object in this paragraph is to obtain for a two-dimensional Poisson-Voronoi tessellation the precise estimate of the logarithmic equivalent of the Laplace transform of the distribution of the square of the fundamental frequency (that means the first eigenvalue λ_1 of the Dirichlet-Laplacian) of the typical cell. This result yields by a Tauberian argument, the asymptotic when $t\to 0^+$ of the logarithm of the distribution function $\mathbf{P}\{\lambda_1\le t\}$.

Proof of (6). Observe that according to the obvious inequalities

$$e^{-t\mu_1} \le e^{-t\lambda_1} \le \varphi(t), \quad t>0,$$

it suffices to prove that

$$\lim_{t\to+\infty} t^{-1/2}\ln \mathbf{E}e^{-t\mu_1} = -4\sqrt{\pi}j_0 \tag{36}$$

and

$$\limsup_{t\to+\infty} t^{-1/2}\ln\mathbf{E}\varphi(t) \le -4\sqrt{\pi}j_0. \tag{37}$$

In order to obtain the asymptotic (36), let us note that the random radius R_m of the largest disc centered at the origin and included in $C(0)$ has the distribution

$$\mathbf{P}\{R_m\ge r\} = e^{-4\pi r^2}, \quad r\ge 0.$$

Thus using the fact that the first eigenvalue of a disc of radius $r > 0$ is equal to j_0^2/r^2 it comes that

$$\begin{aligned}\mathbf{E}e^{-t\mu_1} &= \int_0^{+\infty} 8\pi r e^{-t\frac{j_0^2}{r^2}-4\pi r^2} dr \\ &= t^{1/2}\int_0^{+\infty} 8\pi r \exp\{-t^{1/2}(\frac{j_0^2}{r^2}+4\pi r^2)\}dr,\end{aligned}$$

and by Laplace method we obtain (36).

To prove (37) is more difficult. First, by Theorem 1 we have

$$\begin{aligned}\mathbf{E}\varphi(t) &= \frac{1}{2\pi}\int \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx \\ &= \frac{1}{2\pi}\left[\int_{\{||x||\le t\}} \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx + \int_{\{||x||>t|\}} \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx\right]\end{aligned}$$

Next, the obvious inequality

$$V_2(\widehat{W},x) \ge \pi||x||^2, \quad x\in\mathbb{R}^d,$$

implies that

$$\begin{aligned}\int_{\{||x||>t\}} \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx &\le \int_{\{||x||>t\}} e^{-2t\pi||x||^2}dx \\ &= \left(\frac{\pi}{2t}\right)e^{-2\pi t^3}, \quad t>0.\end{aligned}$$

Therefore

$$\lim_{t\to+\infty} t^{-1/2}\ln\int_{\{||x||>t\}} \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx = -\infty,$$

and consequently in order to obtain (37) it suffices to prove that

$$\limsup_{t\to+\infty} t^{-1/2}\ln\int_{\{||x||\le t\}} \overline{\mathbf{E}}e^{-2tV_2(\widehat{W},x)}dx \le -4\sqrt{\pi}j_0. \tag{38}$$

The key estimations which enable us to derive the above asymptotic estimation are the two following lemmas of independent interest.

Lemma 1 *Denote by $V_1(\widehat{\mathbf{W}})$ the perimeter of the convex hull $\widehat{\mathbf{W}}$ of the two-dimensional Brownian bridge sample path. Fix $0<\varepsilon<1$. There exists $y_0>0$ such that*

$$\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right) \le \exp\left(-2\pi^2{j_0}^2(1-\varepsilon)^3/y^2\right), \quad \forall\, 0\le y\le y_0.$$

Lemma 2 *For $t>0$, and $x\in\mathbb{R}^d$,*

$$\overline{\mathbf{E}}e^{-2tV_d(\widehat{\mathbf{W}},x)} \le \overline{\mathbf{E}}e^{-2tV_d(\widehat{\mathbf{W}},0)}. \tag{39}$$

We will prove Lemma 1 and Lemma 2 at the end of the section. To complete the proof of (6) let us notice that Lemma 2 implies the inequality

$$\int_{\{||x||\le t\}} \overline{\mathbf{E}} e^{-2tV_2(\widehat{W},x)}dx \le \pi t^2 \overline{\mathbf{E}} e^{-2tV_2(\widehat{W},0)}, \quad t>0.$$

Therefore in order to prove (38), it suffices to have

$$\limsup_{t\to+\infty} t^{-1/2}\ln \overline{\mathbf{E}} e^{-2tV_2(\widehat{W},0)} \le -4\sqrt{\pi} j_0. \tag{40}$$

By formula (12), the Cauchy-Schwarz inequality and the Cauchy formula giving the perimeter of a planar convex set we get

$$\begin{aligned} V_2(\widehat{\mathbf{W}},0) &= 2\int_{\mathbb{S}^1}\left(\sup_{z\in\widehat{\mathbf{W}}}(z\cdot u)\right)^2 d\nu_2(u) \\ &\ge \frac{1}{\pi}\left(\int_{\mathbb{S}^1}\sup_{z\in\widehat{\mathbf{W}}}(z\cdot u)\ d\nu_2(u)\right)^2 \\ &= \frac{1}{\pi}\left[V_1(\widehat{\mathbf{W}})\right]^2. \end{aligned} \tag{41}$$

Hence for $y_0>0$, we obtain

$$\begin{aligned} \overline{\mathbf{E}} e^{-2tV_2(\widehat{\mathbf{W}},0)} &\le \overline{\mathbf{E}} e^{-\frac{2t}{\pi}V_1(\widehat{\mathbf{W}})^2} \\ &= \frac{4t}{\pi}\int_0^{+\infty}\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right)e^{-\frac{2t}{\pi}y^2}ydy \\ &= \frac{4t}{\pi}\left[\int_0^{y_0}\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right)e^{-\frac{2t}{\pi}y^2}ydy + \int_{y_0}^{+\infty}\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right)e^{-\frac{2t}{\pi}y^2}ydy\right] \\ &\le e^{-\frac{2t}{\pi}{y_0}^2} + \frac{4t}{\pi}\int_0^{y_0}\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right)e^{-\frac{2t}{\pi}y^2}ydy \end{aligned} \tag{42}$$

Let us fix now $0<\varepsilon<1$ and let y_0 be as in Lemma 1. Then

$$\begin{aligned} \int_0^{y_0}\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}})\le y\right)e^{-\frac{2t}{\pi}y^2}ydy &\le \int_0^{y_0} e^{-\frac{2\pi^2 j_0^2(1-\varepsilon)^3}{y^2}}e^{-\frac{2t}{\pi}y^2}ydy \\ &\le \int_0^{+\infty} e^{-\sqrt{t}\left(\frac{2}{\pi}y^2+\frac{2\pi^2{j_0}^2(1-\varepsilon)^3}{y^2}\right)}ydy, \end{aligned}$$

and by Laplace method

$$\int_0^{+\infty} e^{-\sqrt{t}\left(\frac{2}{\pi}y^2+\frac{2\pi^2{j_0}^2(1-\varepsilon)^3}{y^2}\right)}ydy \sim \frac{K}{t^{1/4}}e^{-4j_0\sqrt{\pi}(1-\varepsilon)^{3/2}t^{1/2}}, \quad \text{when } t\to+\infty, \tag{43}$$

where K is a positive constant.

Combining (42) and (43) we get (40) and (37). So the proof of (6) is now completed.

$\square$

Proof of Lemma 1. The proof is based on the estimate precising the asymptotic behaviour of the Laplace transform of $V_1(\widehat{\mathbf{W}})$ (see [7]):

$$\lim_{t\longrightarrow\infty} t^{-2/3}\ln \overline{\mathbf{E}}\exp(-tV_1(\widehat{\mathbf{W}})) = -\left(\frac{27}{2}\pi^2 {j_0}^2\right)^{1/3}. \tag{44}$$

Let $a = \left(\frac{27}{2}\pi^2 {j_0}^2\right)^{1/3}$ and fix $0 < \varepsilon < 1$. According to (44), choose $u_0 > 0$ such that

$$\overline{\mathbf{E}}\exp(-uV_1(\widehat{\mathbf{W}})) \le e^{-(1-\varepsilon)au^{2/3}}, \quad \forall u \ge u_0.$$

Therefore applying the Tchebychev inequality we obtain the estimation

$$\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}}) \le y\right) \le e^{-(1-\varepsilon)au^{2/3}+uy}, \tag{45}$$

valid for all $u \ge u_0$ and all $y \ge 0$. Now, choosing

$$u = (2(1-\varepsilon)a/3)^3/y^3, \quad 0 < y \le y_0$$

with

$$y_0 = (2(1-\varepsilon)a/(3{u_0}^{1/3})$$

and inserting this in (45) we find

$$\overline{\mathbf{P}}\left(V_1(\widehat{\mathbf{W}}) \le y\right) \le \exp\left\{-2\pi^2 j_0^2(1-\varepsilon)^3/y^2\right\}, \quad 0 \le y \le y_0,$$

which proves Lemma 1.

$\square$

Proof of Lemma 2. Let us fix $x \in \mathbb{R}^d$. The identity

$$\overline{\mathbf{E}}e^{-2tV_d(\widehat{\mathbf{W}},x)} = 2t\int_0^\infty \overline{\mathbf{P}}\left(V_d(\widehat{\mathbf{W}},x) \le u\right)e^{-2tu}du, \quad t > 0,$$

tells us that in order to prove Lemma 2, it suffices to show that

$$\overline{\mathbf{P}}\left(V_d(\widehat{\mathbf{W}},x) \le u\right) \le \overline{\mathbf{P}}\left(V_d(\widehat{\mathbf{W}},0) \le u\right), \quad u \ge 0. \tag{46}$$

Note that the set $B(\widehat{\mathbf{W}};0)$ is not included in $B(\widehat{\mathbf{W}};x)$ so the above inequality is far from obvious.

We need the following lemma:

Lemma 3 *Fix $N \ge 1$ and $u \ge 0$. Then the set*

$$A_u = \left\{z = (z_1,\cdots,z_N) \in (\mathbb{R}^d)^N; V_2\left(\bigcup_{i=1}^N B(z_i,||z_i||)\right) \le u\right\}, \quad d \ge 2,$$

is convex and symmetric.

Proof. Clearly A_u is symmetric. To prove that A_u is convex, fix

$$\overline{z} = (z_1, \cdots, z_N) \in A_u, \quad \overline{y} = (y_1, \cdots, y_N) \in A_u, \quad 0 < \gamma < 1,$$

and define

$$\overline{v} = (v_1, \cdots, v_N) = (1-\gamma)z + \gamma y.$$

We note also $z_0 = y_0 = v_0 = 0$.

From (12), we obtain

$$\begin{aligned}
&V_d\left(\bigcup_{i=1}^{N} B(v_i, ||v_i||)\right) \\
&= \frac{2^d}{d} \int_{\mathbb{S}^{d-1}} \left[\sup_{i=0,\cdots,N} (v_i \cdot u)\right]^d d\nu_d(u) \\
&= \frac{2^d}{d} \int_{\mathbb{S}^{d-1}} \left[\sup_{i=0,\cdots,N} (\gamma z_i \cdot u + (1-\gamma) y_i \cdot u)\right]^d d\nu_d(u) \\
&\leq \frac{2^d}{d} \left[\gamma \int \left(\sup_{i=0,\cdots,N} (z_i \cdot u)\right)^d d\nu_d(u) + (1-\gamma) \int \left(\sup_{i=0,\cdots,N} (y_i \cdot u)\right)^d d\nu_d(u)\right] \\
&= \gamma V_d\left(\bigcup_{i=1}^{N} B(z_i, ||z_i||)\right) + (1-\gamma) V_d\left(\bigcup_{i=1}^{N} B(y_i, ||y_i||)\right) \\
&\leq u
\end{aligned}$$

which proves the lemma.

□

We are now ready to prove the inequality (46). In order to do it, let us fix $N \geq 2$ ans select $0 \leq t_1 \leq \cdots \leq t_N \leq 1$. The law of the random vector

$$\mathcal{W} = (W(t_1), \cdots, W(t_N)) \in (\mathbb{R}^d)^N$$

is a centered Gaussian measure. Let us observe that

$$\overline{\mathbf{P}}\left\{V_d\left(\bigcup_{i=1}^{N} B(W(t_i), ||W(t_i) - x||)\right) \leq u\right\} = \overline{\mathbf{P}}\left\{\mathcal{W} \in A_u + \overline{x}\right\} \tag{47}$$

with $\overline{x} = (x, \cdots, x) \in (\mathbb{R}^d)^N$.

The set A_u, $u > 0$, being convex and symmetric we may apply Anderson's lemma [1] which gives

$$\overline{\mathbf{P}}\left\{\mathcal{W} \in A_u + \overline{x}\right\} \leq \overline{\mathbf{P}}\left\{\mathcal{W} \in A_u\right\}. \tag{48}$$

Next take a countable set $(t_i)_{i\geq 1}$ dense in $[0,1]$. By the continuity of the Brownian bridge sample paths, the set $D = (W(t_i))_{i\geq 1}$ is a dense subset of W. Then by (9), we have

$$V_d(D, x) = V_d(\mathbf{W}, x) = V_d\left(\overline{\cup_{y\in D} B(y, ||y - x||)}\right), \quad x \in \mathbb{R}^d.$$

Thus the increasing sequence $V_d\left(\cup_{i=1}^N B(y_i, ||y_i - x||)\right)$, $N \geq 1$, converges to $V_d(\mathbf{W}, x)$ for all $x \in \mathbb{R}^d$, which implies

$$\overline{\mathbf{P}}\left\{V_d\left(\widehat{\mathbf{W}}, x\right) \leq u\right\} \quad = \quad \lim_{N\to+\infty} \overline{\mathbf{P}}\left\{V_d\left(\bigcup_{i=1}^N B(W(t_i), ||W(t_i) - x||)\right) \leq u\right\}, \quad u \geq 0.$$

Combining this with (48), we obtain the inequality (46) and the proof of Lemma 2 is completed.

□

Proof of (7). Let us recall first that

$$\lambda_1 \leq \mu_1 = \frac{{j_0}^2}{{R_m}^2},$$

the (random) radius R_m of the largest disc contered at the origin and included in $\mathcal{C}$ having the law

$$\mathbf{P}\{R_m \geq u\} = \exp(-4\pi u^2), \quad u \geq 0.$$

Hence

$$-4\pi {j_0}^2 = \lim_{t\to 0^+} t \ln \mathbf{P}\{\mu_1 \leq t\} \leq \limsup_{t\to 0^+} \ln \mathbf{P}\{\lambda_1 \leq t\}.$$

The upper bound

$$\limsup_{t\to 0^+} t \ln \mathbf{P}\{\lambda_1 \leq t\} = -4\pi {j_0}^2$$

follows easily from the asymptotic (6) by applying the method described in the proof of Lemma 1.

□

5 Concluding remarks.

(1) A part of the above arguments works in a more general setting of Johnson-Mehl tessellation [3], [10], [16]. In that model the crystals start growing radially (in all direction at fixed speed $v > 0$) at time t_i from the nuclei x_i in such a way that

$$\Phi = \{(x_i, t_i) \in \mathbb{R}^d \times [0, +\infty)\}$$

is a spatially-homogeneous Poisson point process. At the end of growth the whole space is covered and the construction of the Johnson-Mehl tessellation is completed. The Poisson-Voronoi tessellation corresponds to the particular case when all nuclei are born at the same time. The Johnson-Mehl crystals are star-shaped but not necessarily convex, the common boundary between two crystals, which is a part of a hyperboloid, may even be disconnected.

It can be shown that for a process Φ satisfying the canonical conditions of J. Møller [16] the expectation of the spectral function of the typical Johnson-Mehl cell can also be expressed in terms of Brownian bridge from which a two-terms expansion near the origin can be obtained (see [8]).

(2) It would be interesting to obtain the geometric significance of the coefficients $c_{d,2}$ appearing in the asymptotic of Theorem 2. In view of (24) it is likely that $c_{d,2}$ is connected with

$$\mathbf{E}\left\{\sum_{i=1}^{N_{d-2}(\mathcal{C})}\frac{1}{3}\left(\frac{\omega_i(\mathcal{C})}{\pi}-\frac{\pi}{\omega_i(\mathcal{C})}\right)V_{d-2}(F_i(\mathcal{C}))\right\},$$

where

(i) $F_i(\mathcal{C})$, $i=1,\cdots,N_{d-2}(\mathcal{C})$ are the $(d-2)$-dimensional faces of $\mathcal{C}$;

(ii) $\omega_i(\mathcal{C})$ is the magnitude of the dihedral angle at the face $F_i(\mathcal{C})$, $1\leq i\leq N_{d-2}(\mathcal{C})$.

(3) To obtain the values of coefficients $c_{d,i}$, $i\geq 3$, in (25) it is necessary to calculate explicitely the covariances $\overline{\mathbf{E}}M_{u_1}\cdots M_{u_j}$, $u_1,\cdots,u_j\in\mathbb{S}^{d-1}$, $j\geq 3$, associated to the d-dimensional Brownian bridge. At our knowledge this problem is open. Note also that for a bounded convex polyhedron of $\mathbb{R}^d$ the explicit expressions of the coefficients at order $k\geq 4$ appearing in the asymptotic (near the origin) of the spectral function are unknown (see [4]).

(4) The method used to prove Theorem 3 works in any dimension $d\geq 2$. Meanwhile to get the final result it is necessary to know the value of the logarithmic equivalent (at infinity) of the Laplace transform of the mean width of $\widehat{\mathbf{W}}$. The mean width of $\widehat{\mathbf{W}}$ is connected [7] with the d-dimensional measure of the empirical cell $\widetilde{\mathcal{C}}$ of the Poisson hyperplane tessellation introduced by R. E. Miles. In dimension $d=2$ the asymptotic of the area of the cell $\widetilde{\mathcal{C}}$ is known (see [7],[12]) from which the estimation (44) follows. Unfortunately no such result is known in dimension $d\geq 3$. This prevents us to extend our result to the case $d\geq 3$.

(5) It is interesting to note the significance of Lemma 2. The inequality (29) can be rewritten under the form

$$\mathbf{E}\overline{\mathbf{P}^{\mathbf{x}}}\{T>u\}\leq\mathbf{E}\overline{\mathbf{P}^{\mathbf{0}}}\{T>u\},\quad u\geq 0,$$

where T denotes the first exit time of the Brownian bridge from the cell $C(0)$, the notation $\overline{\mathbf{P}^{\mathbf{x}}}$ expressing the fact that the Brownian path is starting at the point $x\in\mathbb{R}^d$. If we replace the Brownian bridge by the standard Brownian motion in $\mathbb{R}^d$ in the proof of Theorem 2 we obtain the corresponding inequality

$$\mathbf{E}\overline{\mathbf{P}^{\mathbf{x}}}\{\tau>u\}\leq\mathbf{E}\overline{\mathbf{P}^{\mathbf{0}}}\{\tau>u\},\quad u\geq 0,$$

for the first Brownian exit time τ of $C(0)$.

(6) It is well known that "the larger regions have smaller eigenvalues" and therefore the equalities (6) and (7) express that in some sense the large Voronoi cells are nearly circular. An analogous phenomenon (known under the name of D. G. Kendall conjecture) occurs for the polygons determined by a standard Poisson line process in the plane (see [7], [12]).

Appendix. Proof of (35).

Let us recall that by associating to each vertex s of the planar Poisson-Voronoi tessellation the triangle $T(s)$ whose vertices are the nuclei of the three cells containing s (and whose center of circumdisc coincides precisely with s), we obtain the dual tessellation, called the Delaunay tessellation (see [17]). The typical Delaunay cell $\mathcal{D}$ is defined (in Palm sense) by the following formula [17]:

$$\mathbf{E}h(\mathcal{D}) = \frac{1}{\lambda_0 V_2(B)}\mathbf{E}\sum_{s\in\mathcal{S}\cap B} h(T(s)-s), \tag{49}$$

for all measurable function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$, and where:

(i) $B \subset \mathbb{R}^2$ is an arbitrary fixed Borel set such that $0 < V_2(B) < +\infty$;

(ii) $\mathcal{S}$ is the set of vertices of the Poisson-Voronoi tessellation.

Note that the mean number of vertices per unit area is equal to 2 (see [17]).

The distribution of the typical cell $\mathcal{D}$, which is a triangle noted $z_1z_2z_3$, is known explicitely by means of the distributions of the radius $\rho(\mathcal{D})$ of the circumdisc of $\mathcal{D}$ and of the three angles $\beta_1(\mathcal{D}) = \widehat{z_1pz_2}$, $\beta_2(\mathcal{D}) = \widehat{z_2pz_3}$, $\beta_3(\mathcal{D}) = \widehat{z_1pz_2}$, where p is the center of the circumdisc of $\mathcal{D}$ (see [17], p. 104, for an expression of these distributions valid in any dimension and [19], p. 249, for a rewriting in dimension 2). In particular, these angles are identical in law and independent of $\rho(\mathcal{D})$. This implies that the angles $\alpha_1(\mathcal{D}), \alpha_2(\mathcal{D}), \alpha_3(\mathcal{D})$ spanned in p by any two edges of the Voronoi tessellation, are independent of $\rho(\mathcal{D})$ and of common distribution (see for example [19]):

$$\alpha_1(\mathcal{D})(P)(dt) = \frac{4}{3\pi}\sin t(\sin t - t\cos t)1_{[0.\pi]}(t)dt. \tag{50}$$

Thus from (50) we get the following result:

Lemma 4 *We have:*

$$\mathbf{E}\frac{1}{\alpha_1(\mathcal{D})} = \frac{2}{3\pi}\int_0^{2\pi}\frac{1-\cos t}{t}dt.$$

Let us consider now the set $\mathcal{A}$ of angles of the Poisson-Voronoi tessellation and for each $\alpha \in \mathcal{A}$, let $s(\alpha)$ be the associated vertex. Then noticing that the mean number of angles per unit area is equal to six we can define a typical angle $\overline{\alpha}$ on the following lines:

$$\mathbf{E}1_{[0,t]}(\overline{\alpha}) = \frac{1}{6V_2(B)}\mathbf{E}\sum_{\substack{\alpha\in\mathcal{A}\\ s(\alpha)\in B}} 1_{[0,t]}(\alpha), \quad 0 \le t \le \pi, \tag{51}$$

where B is a fixed Borel set of $\mathbb{R}^2$ such that $0 < V_2(B) < \infty$.

The typical angle $\overline{\alpha}$ can be connected, on one hand to the angles of the Voronoi typical cell $\mathcal{C}$, and on the other hand to the angles $\alpha_1(\mathcal{D}), \alpha_2(\mathcal{D}), \alpha_3(\mathcal{D})$ of the Delaunay typical cell $\mathcal{D}$. More precisely, we have:

Lemma 5 **(i)** *The angles $\overline{\alpha}$ et $\alpha_1(\mathcal{D})$ are identical in law.*

(ii) *for any measurable function $f : [0,\pi] \longrightarrow \mathbb{R}_+$, we have*

$$\mathbf{E}f(\overline{\alpha}) = \frac{1}{6}\mathbf{E}\sum_{i=1}^{N_0(\mathcal{C})} f(\alpha_{\mathcal{C},i}),$$

where $\alpha_{\mathcal{C},i}$, $1 \le i \le N_0(\mathcal{C})$, denote the inside-facing angles of the typical cell $\mathcal{C}$.

Proof. **(i)** Consider $0 \le t \le \pi$ and $B \in \mathcal{B}(\mathbb{R}^2)$, $V_2(B) = 1$.
Then by definition (49) of $\mathcal{D}$, we have

$$\frac{1}{3}\mathbf{E}\sum_{i=1}^{3} 1_{[0,t]}(\alpha_i(\mathcal{D})) = \frac{1}{6}\mathbf{E}\sum_{s\in\mathcal{S}\cap B}\left(\sum_{i=1}^{3} 1_{[0,t]}(\alpha_i(s))\right)$$

where $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$ are the three angles associated with the vertex s in the tessellation.
Moreover, by definition of the typical angle (see (51)),

$$\frac{1}{6}\mathbf{E}\sum_{s\in\mathcal{S}\cap B}\left(1_{[0,t]}(\alpha_1(s)) + 1_{[0,t]}(\alpha_2(s)) + 1_{[0,t]}(\alpha_3(s))\right) = \mathbf{E}1_{[0,t]}(\overline{\alpha}).$$

We conclude then by using the identity in law of the three angles $\alpha_1(\mathcal{D})$, $\alpha_2(\mathcal{D})$ and $\alpha_3(\mathcal{D})$.
(ii) The proof is similar to that of Prop. 3.2.2. of J. Møller [17] which connects, for $1 \le k \le d$,, the k-dimensional typical face of the Voronoi tessellation to the k-dimensional faces of the typical cell.

$\square$

From Lemmas 4 and 5, we deduce that

$$\mathbf{E}\alpha^{-1}(\mathcal{C}) = 6\mathbf{E}\left(\frac{1}{\overline{\alpha}}\right) = 6\mathbf{E}\left(\frac{1}{\alpha_1(\mathcal{D})}\right) = \frac{4}{\pi}\int_0^{2\pi}\frac{1-\cos t}{t}dt,$$

which completes the proof of (35).

$\square$

References

[1] T. W. Anderson. The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.*, 6:170–176, 1955.

[2] F. Baccelli and B. Blaszczyszyn. On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Rapport de recherche INRIA No 4019*, October 2000.

[3] R. Cowan, S. N. Chiu, and L. Holst. A limit theorem for the replication time of a DNA molecule. *J. Appl. Probab.*, 32(2):296–303, 1995.

[4] B. V. Fedosov. Asymptotic formulae for eigenvalues of the Laplace operator for a polyhedron. *Dokl. Akad. Nauk SSSR*, 157:536–538, 1964.

[5] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33:958–972, 1962.

[6] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[7] A. Goldman. Sur une conjecture de D. G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne. *Ann. Probab.*, 26(4):1727–1750, 1998.

[8] A. Goldman and P. Calka. On the spectral function of the Johnson-Mehl and Voronoi tessellations. *Preprint 00-02 of LaPCS*, 2000.

[9] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[10] W. A. Johnson and R. F. Mehl. Reaction kinetics in processes of nucleation and growth. *Trans. Amer. Inst. Min. Engrs.*, 135:416–458, 1939.

[11] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[12] I. N. Kovalenko. A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons. *J. Appl. Math. Stochastic Anal.*, 12(4):301–310, 1999.

[13] S. Kumar and S. K. Kurtz. A Monte-Carlo study of size and angular properties of a three dimensional Poisson-Delaunay cell. *J. Stat. Phys.*, 75:735–748, 1993.

[14] S. Kumar and R. N. Singh. Thermal conductivity of polycristalline materials. *J. of the Amer. Cer. Soc.*, 78(3):728–736, 1995.

[15] J. L. Meijering. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.*, 8, 1953.

[16] J. Møller. Random Johnson-Mehl tessellations. *Adv. in Appl. Probab.*, 24(4):814–844, 1992.

[17] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[18] J. Møller. Generation of Johnson-Mehl crystals and comparative analysis of models for random nucleation. *Adv. in Appl. Probab.*, 27(2):367–383, 1995.

[19] L. Muche. The Poisson Voronoi tessellation. III. Miles' formula. *Math. Nachr.*, 191:247–267, 1998.

[20] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams.* John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[21] E. Pielou. *Mathematical ecology.* Wiley-Interscience, New-York, 1977.

[22] M. H. Protter. Can one hear the shape of a drum? revisited. *SIAM Rev.*, 29(2):185–197, 1987.

[23] R. T. Rockafellar. *Convex analysis.* Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 28.

[24] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications.* John Wiley & Sons Ltd., Chichester, 1987. With a foreword by D. G. Kendall.

[25] D. W. Stroock. *Probability theory, an analytic view.* Cambridge University Press, Cambridge, 1993.

[26] R. van de Weygaert. Fragmenting the Universe III. The construction and statistics of 3-D Voronoi tessellations. *Astron. Astrophys.*, 283:361–406, 1994.

[27] M. van den Berg and S. Srisatkunarajah. Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary. *J. London Math. Soc. (2)*, 37(1):119–127, 1988.

[28] H. Weyl. Über dis asymptotische verteilung der Eigenwerte. *Göttinger Nachr.*, pages 110–117, 1911.

Chapitre 4

Quelques conséquences de l'identité entre cellule typique et cellule empirique pour la mosaïque poissonienne d'hyperplans d-dimensionnelle.

4.1 Présentation des résultats.

Dans toute cette partie, Φ désignera un processus ponctuel de Poisson de mesure d'intensité μ, $\mathcal{C}$ la cellule typique de la mosaïque poissonienne d'hyperplans d-dimensionnelle, et Ψ le processus ponctuel (stationnaire) constitué des centres des disques inscrits dans les cellules de la mosaïque. On notera σ_d l'aire de la boule-unité de $\mathbb{R}^d$ et $B(y,r)$ la boule de centre $y \in \mathbb{R}^d$ et de rayon $r \geq 0$.

Rappelons que nous avons montré que la cellule empirique peut être interprétée en terme de mesure de Palm :

pour toute application $h : \mathcal{K} \longrightarrow \mathbb{R}$ mesurable bornée,

$$\mathbf{E}h(\mathcal{C}) = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \frac{1}{V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(C(z) - z),$$

où $B \subset \mathbb{R}^d$ est un borélien fixé vérifiant $0 < V_d(B) < +\infty$.

La définition au sens de Palm autorise en particulier l'utilisation de certains résultats qui n'ont jamais été exploités dans ce contexte. On peut citer notamment la caractérisation de la mesure de Lebesgue de $\mathbb{R}^d$ comme seule mesure invariante par toute translation (à coefficient multiplicatif prés) et la la formule de Slivnyak (voir le paragraphe 1.2).

Rappelons qu'elle a permis en particulier dans le cas de la mosaïque de Poisson-Voronoi de réaliser explicitement la cellule typique comme la cellule $C(0)$ associée à un germe placé en l'origine lorsque l'on rajoute ce germe au processus de départ. Ce fait a été déterminant pour obtenir des informations sur la cellule typique (voir les chapitres

2 et 3). De la même manière, on peut esérer de réaliser concrètement la cellule typique d'une mosaïque poissonienne d'hyperplans en utilisant cette formule.

R. E. Miles a déjà obtenu en 1973 une construction explicite de la cellule empirique en dimension deux à partir de son disque inscrit et de son triangle circonscrit, dont les lois exactes sont connues [59]. Sa méthode reposant sur la notion de circuit convexe dans le plan, semble difficile à généraliser en dimension supérieure.

Dans le travail qui suit, on détermine un moyen concret de réaliser la cellule typique. Considérons une boule $B(0, R_I)$ où R_I suit la loi exponentielle de paramètre σ_d. La cellule $\mathcal{C}$ est alors égale en loi à l'intersection de :

(i) le simplexe circonscrit à cette boule et indépendant de R_I tel que la densité de la loi conjointe des vecteurs unitaires orthogonaux aux hyperfaces est proportionnelle au volume du simplexe formé par ces vecteurs ;

(ii) la cellule de Crofton de la mosaïque poissonienne construite à partir d'un processus ponctuel de Poisson de mesure d'intensité $\mathbf{1}_{B(0,r)^c} d\mu$ conditionnellement à $\{R_I = r\}$, $r \geq 0$.

Remarquons que la loi du simplexe circonscrit à $\mathcal{C}$ rappelle celle de la cellule typique d'une mosaïque de Poisson-Delaunay d-dimensionnelle (voir [63],[65]).

On déduit de ce premier résultat une formule explicite pour $\mathbf{P}\{N_{d-1}(\mathcal{C}) = d + 1\}$, où $N_{d-1}(\mathcal{C})$ désigne le nombre d'hyperfaces de $\mathcal{C}$. On généralise ainsi en toute dimension les calculs effectués par Miles en dimensions deux et trois [56].

Par ailleurs, en utilisant le résultat de Miles selon lequel une section par un sous-espace affine de $\mathbb{R}^d$ d'une mosaïque poissonienne reste une mosaïque poissonienne à une homothétie près [56], la formule de Slivnyak nous permet d'une part de retrouver le calcul de la moyenne de l'"aire" k-dimensionnelle des k-faces de $\mathcal{C}$, $0 \leq k \leq d$, obtenu par G. Matheron [48] en 1975, et d'autre part de déterminer le premier moment du nombre de faces k-dimensionnelles de la cellule typique. Ce dernier résultat n'était connu jusquà présent qu'en dimensions deux et trois (voir [56]).

Poissonian tessellations of the Euclidean space. An extension of a result of R. E. Miles. *

Pierre Calka†

Abstract

In 1973, R. E. Miles obtained an explicit characterization of the typical cell of the planar Poissonian tessellation, by means of the distributions of the indisc and the triangle circumscribed to the cell. In this paper, we propose a different proof, using Xthe classical formula of Slivnyak for Poisson point processes. Not only the method is simple and rigorous, but it also extends the result of Miles to any dimension $d \geq 2$. We deduce from it some other properties of the geometrical characteristics of the typical cell.

Introduction.

Let Φ be a Poisson point process in $\mathbb{R}^d$, $d \geq 2$, of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r, u) d\nu_d(u) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where ν_d is the area measure of the unit-sphere $\mathbb{S}^{d-1}$.

Let us consider for all $x \in \mathbb{R}^d$, $H(x) = \{y \in \mathbb{R}^d; (y-x) \cdot x = 0\}$, ($x \cdot y$ being the usual scalar product). Then the set $\mathcal{H} = \{H(x); x \in \Phi\}$ divides the space into convex polyhedra that constitute the so-called *d-dimensional Poissonian tessellation.* This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.

This random object was used for the first time by S. A. Goudsmit [7] and by R. E. Miles ([10], [11] and [13]). In particular, it provides a model for the fibrous structure of sheets of paper.

Miles introduced in particular the notion of *empirical* (or *typical*) cell associated to the tessellation. Recent contributions about the law of the area of the typical cell and the famous D. G. Kendall conjecture were provided by A. Goldman [5] and I. N. Kovalenko [8]. The fundamental frequencies of the cells have been studied by A. Goldman [4] and

**American Mathematical Society 2000 subject classifications.* Primary 60D05; secondary 60G55.
Key words and phrases. Empirical distributions, Palm distribution, Poissonian tessellations, stochastic geometry. *Running head.* Poissonian tessellations.

†*Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr

have been useful to obtain informations about the frequencies of the cells of the Poisson-Voronoi tessellation [6]. Central limit theorems have been provided by K. Paroux [17] in this context. Besides, we have obtained the explicit distribution of the radius of the smallest disk centered at the origin containing the Crofton cell in the plane [2].

Miles obtained [13] in the two-dimensional case the explicit distributions of the indisk radius and the circumscribed triangle of the typical cell, which has provided numerous results concerning the area, the perimeter and the number of vertices. His method (see [13]) essentially relies on the concept of convex circuit in $\mathbb{R}^2$ and the ergodic properties of the tessellation (see also [3]); Nevertheless, it can not be generalized to any dimension.

In this work, we propose a new approach which is easier and can be extended to any dimension $d \geq 2$. It consists to use famous Slivnyak's formula which is a fundamental tool for the study of Poisson-Voronoi tessellations [15]. Nevertheless, to use this formula in our context, we need to describe the typical cell of R. E. Miles with a Palm procedure [16]. In the first section, we actually obtain such a description via the (stationary) point process of the centers of inballs of the cells constituting the tessellation. We then prove in the second section the principal result concerning the construction of the typical cell via the law of the inball radius and the circumscribed simplex. In the last section, we show how to obtain some new informations about the geometric characteristics of the typical cell by using Slivnyak's formula.

All the results presented here were announced in a previous note [1].

1 Preliminaries.

Let Φ be a Poisson point process in $\mathbb{R}^d$ of intensity measure

$$\mu(A) = \mathbf{E} \sum_{x \in \Phi} \mathbf{1}_A(x).$$

The process Φ is a measurable application which takes values in the space $\mathcal{M}_\sigma$ of the locally finite sets of $\mathbb{R}^d$ endowed with the cylindric σ-field $\mathcal{T}_c$ generated by the applications

$$\varphi_A : \left\{ \begin{array}{lcl} \mathcal{M}_\sigma & \longrightarrow & \mathbb{N} \cup \{+\infty\} \\ \gamma & \longmapsto & \#(A \cap \gamma) \end{array} \right. , \quad A \in \mathcal{B}(\mathbb{R}^d).$$

For any positive measurable function f on the product space $(\mathbb{R}^d)^n \times \mathcal{M}_\sigma$, $n \in \mathbb{N}^*$, which is invariant by permutation of the first n coordinates, we have Slivnyak's formula (see for example [16]):

$$\mathbf{E} \sum_{\xi \in \overline{\Phi}^{(n)}} f(\xi, \Phi) = \frac{1}{n!} \int \mathbf{E} f\left(\xi, \Phi \cup \xi\right) d\overline{\mu}^{(n)}(\xi), \tag{1}$$

where $\overline{\Phi}^{(n)}$ denotes the space of sets of Φ with cardinal n, and where

$$d\overline{\mu}^{(n)}(\xi) = d\mu(\xi_1) \cdots d\mu(\xi_n), \quad \xi = \{\xi_1, \cdots \xi_n\} \in \overline{\Phi}^{(n)}.$$

Let us now take the measure

$$\mu(A) = \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r, u) d\nu_d(u), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{2}$$

and consider the associated *Poissonian tessellation* [12]. We will note C_0 the cell of the tessellation containing the origin. We show [4] that the cell C_0 (called the *Crofton cell*) is a.s. well-defined. Let us denote by $\mathcal{C}_R$ the set of cells included in the open ball $B(R)$ centered at the origin, with radius $R > 0$ and $N_R = \#\mathcal{C}_R$. Besides, we consider the space $\mathcal{K}$ of the convex compact sets of $\mathbb{R}^d$ endowed with the usual Hausdorff topology and $h : \mathcal{K} \longrightarrow \mathbb{R}$ a translation-invariant, bounded measurable function.

We have the following result [4]:

Theorem 1 *The means*

$$\frac{1}{N_R} \sum_{C \in \mathcal{C}_R} h(C), \quad R > 0, \tag{3}$$

converge a.s. to a constant $\widetilde{\mathbf{E}}h$ *(the empirical mean) satisfying*

$$\widetilde{\mathbf{E}}h = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right),$$

where V_d denotes the d-dimensional Lebesgue measure.

Let us define Ψ as the point process constituted with the centers of the inballs of the cells of the tessellation. The invariance by any translation of the Poissonian tessellation [4] implies that Ψ is a stationary (locally finite) point process. We fix a Borel set $B \subset \mathbb{R}^d$ verifying $0 < V_d(B) < +\infty$. The typical cell $\mathcal{C}$, in the Palm sense, then is defined by the following formula:

$$\mathbf{E}h(\mathcal{C}) = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \frac{1}{V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(C(z) - z), \tag{4}$$

where $h : \mathcal{K} \longrightarrow \mathbb{R}$ runs throughout the set of bounded measurable functions and where ω_d is the Lebesgue measure of the unit-ball of $\mathbb{R}^d$ (the constant $\omega_d {\omega_{d-1}}^d / 2^d$ is the intensity of the process Ψ).

More generally, the definition of $\mathcal{C}$ implies that the following formula, called Campbell's formula, is satisfied:

$$\int \mathbf{E}f(\mathcal{C}, y)dy = \frac{2^d}{\omega_d {\omega_{d-1}}^d} \mathbf{E} \sum_{z \in \Psi} f(C(z) - z, z), \tag{5}$$

for all measurable function $f : \mathcal{K} \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$.
Repeating word for word the method of J. Møller described in [16], page 66 (applying to any stationary point process), we show the identity

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}\left(1/V_d(C_0)\right)} \mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right). \tag{6}$$

Comparing (6) with Theorem 1, we deduce that:

Theorem 2 *For all translation-invariant, bounded measurable function h, we have*

$$\widetilde{\mathbf{E}}h = \mathbf{E}h(\mathcal{C}).$$

2 The principal result.

For all $z \in \Psi$, we note $B(z)$ the inball of the cell $C(z)$ and $S(z)$ the simplex constructed with the support hyperplanes of $C(z)$ which are tangent to $B(z)$. Besides, let us consider

$$\Phi_z = \{x - \frac{x}{||x||^2}(x \cdot z); x \in \Phi, H(x) \cap B(z) = \emptyset\},$$

the point process associated to the hyperplanes which do not intersect $B(z)$ (taking the point z as new origin).
Then, denoting by $C_0(\Phi_z)$ the Crofton cell associated to the hyperplanes $H(y)$, $y \in \Phi_z$, we have easily

$$C(z) = S(z) \cap (z + C_0(\Phi_z)). \tag{7}$$

Let us consider a random couple $(\mathcal{S}, \widehat{\Phi})$, where $\mathcal{S}$ is a (random) simplex and $\widehat{\Phi}$ is a point process such that the joint distribution is given by the formula (analogous to (4)):

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d \omega_{d-1}{}^d V_d(B)} \mathbf{E} \sum_{z \in \Psi \cap B} h(S(z) - z)\mathbf{1}_{\{\Phi_z \cap A = \emptyset\}}, \tag{8}$$

satisfied for every positive measurable function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$, and every fixed Borel set $A \in \mathbb{R}^d$;$B \subset \mathbb{R}^d$ such that $0 < V_d(B) < +\infty$.
Applying (8) to a function $g(\mathcal{S}, \widehat{\Phi}) = h(\mathcal{S} \cap C_0(\widehat{\Phi}))$, where h is translation-invariant, we obtain thanks to (7), the equality

$$\mathbf{E}h(\mathcal{S} \cap C_0(\widehat{\Phi})) = \mathbf{E}h(\mathcal{C}),$$

which means

$$\mathcal{C} \overset{law}{=} \mathcal{S} \cap C_0(\widehat{\Phi}). \tag{9}$$

It then remains to determine the distribution of the couple $(\mathcal{S}, \widehat{\Phi})$.
To this end, let us consider the set $\mathcal{A}$ of all the subsets $\xi \subset \mathbb{R}^d$, $\#\xi = d+1$, such that the associated hyperplanes $H(x)$, $x \in \xi$, form a $(d+1)$-simplex of $\mathbb{R}^d$. For all $\xi \in \mathcal{A}$, let us denote by $S(\xi)$ (resp. $B(\xi)$ and $c(\xi)$) the simplex associated to ξ (resp. the inball of $\mathcal{S}(\xi)$, and the center of this ball).
Besides, we define

$$\mathcal{A}_\Phi = \{\xi \in \overline{\Phi}^{(d+1)} \cap \mathcal{A}; B(\xi) \cap H(x) = \emptyset \ \forall \, x \in \Phi\}.$$

The set $\mathcal{A}_\Phi$ is in one-to-one correspondence with the set of the points of Ψ via the application $\xi \longmapsto c(\xi)$; it is also in one-to-one correspondence with the set $\{S(z); z \in \Psi\}$ via the application S. Using (8), it implies that

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d \omega_{d-1}{}^d V_d(B)} \mathbf{E} \sum_{\xi \in \overline{\Phi}^{(d+1)}} h(S(\xi) - c(\xi))\mathbf{1}_B(c(\xi))\mathbf{1}_{\mathcal{A}_\Phi}(\xi)\mathbf{1}_{\{\Phi_{c(\xi)} \cap A = \emptyset\}}.$$

Applying now Slivnyak's formula (1), we get from the preceding equality

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d \omega_{d-1}{}^d V_d(B)} \frac{1}{(d+1)!} \int h(S(\xi) - c(\xi))\mathbf{1}_B(c(\xi))\mathbf{1}_{\mathcal{A}}(\xi)$$
$$\mathbf{P}\{H(x) \cap B(\xi) = \emptyset, \, \forall x \in \Phi; \Phi_{c(\xi)} \cap A = \emptyset\} d\overline{\mu}^{(d+1)}(\xi) \tag{10}$$

It remains to explicit this integral via appropriated change of variables, and we deduce the principal result:

Theorem 3 **(1)** *Let us consider a ball of center zero and (random) radius R_I of law*

$$\mathbf{P}\{R_I \geq t\} = \exp\{-\sigma_d t\}, \quad t \geq 0,$$

where $\sigma_d = \nu_d(\mathbb{S}^{d-1})$.
(2) *Let us construct a simplex $\mathcal{S}$ circumscribed to this ball such that the $(d+1)$ directions $U_0, \cdots, U_d \in \mathbb{S}^{d-1}$ from the center to the intersecting points are independent from R_I and have a joint distribution given by:*

$$[(U_0, \cdots, U_d)](\mathbf{P})(u) = \frac{d2^d}{(d+1){\sigma_d}^2{\omega_{d-1}}^d}\Delta(U_0, \cdots, U_d)\mathbf{1}_A(u)d\overline{\nu_d}^{(d+1)}(u), \qquad (11)$$

where $d\overline{\nu_d}^{(d+1)}(u) = d\nu_d(u_0)...d\nu_d(u_d)$, $u = (u_0, ..., u_d) \in (\mathbb{S}^{d-1})^{d+1}$,

$$A = \{(u_0, \cdots, u_d) \in (\mathbb{S}^{d-1})^{d+1}; \text{no half-sphere contains } u_0, \cdots, u_d\}, \qquad (12)$$

and $\Delta(x_0, ..., x_d)$, $x_0, ..., x_d \in \mathbb{R}^d$, denotes the d-dimensional Lebesgue measure of the simplex with vertices at $x_0, \cdots, x_d$.
(3) *Let us take a point process $\widehat{\Phi}$ independent from $U_0, \cdots, U_d$ such that conditionally to $R_I = r$, $r > 0$, $\widehat{\Phi}$ is distributed as a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c}d\mu$. Then we have the following equality in law:*

$$\mathcal{C} \overset{law}{=} \mathcal{S} \cap C_0(\widehat{\Phi}).$$

Proof. The following lemma provides a formula of change of variables of Blaschke-Petkantschin type (see for example [14]):

Lemma 1 *We have:*

$$\mathbf{1}_{\mathcal{A}}(\xi)d\overline{\mu}^{(d+1)}(\xi) = d!\Delta(u)\mathbf{1}_A(u)dzdRd\overline{\nu_d}^{(d+1)}(u).$$

Proof of Lemma 1. We first remark that $\mathcal{A}$ is in bijection with $\mathbb{R}^d \times \mathbb{R}_+^* \times A$ (where A is defined in (12)) by associating to each $\xi = \{\xi_0, \cdots, \xi_d\} \in \mathcal{A}$, $z = c(\xi)$, the radius R of $B(\xi)$ and the unit-directions $u_0, \cdots, u_d$ from $c(\xi)$ to $\xi_0, \cdots, \xi_d$ respectively.

Let us define for all $0 \leq i \leq d$, $p_i = ||\xi_i||$ and $v_i = \xi_i/p_i \in \mathbb{S}^{d-1}$ such that

$$d\mu(\xi_i) = \mathbf{1}_{p_i \geq 0}dp_i d\nu_d(v_i).$$

It is easy to prove that

$$p_i = |R + (z \cdot u_i)|, \quad v_i = \begin{cases} u_i & \text{if } ||z|| \leq p_i \\ -u_i & \text{else} \end{cases}$$

So the jacobian of the one-to-one correspondence

$$(z, R, u_0, \cdots, u_d) \longmapsto (p_0, \cdots, p_d, v_0, \cdots, v_d)$$

is

$$J = \begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

where $C = 0$, $D = I_{d(d+1)}$ is the unit-matrix,

$$A = \begin{pmatrix} {u_0}^t & 1 \\ \vdots & \vdots \\ {u_d}^t & 1 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} z^t & 0 & \dots & 0 \\ 0 & z^t & \ddots & \vdots \\ \vdots & & & \\ & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z^t \end{pmatrix}.$$

Consequently, $J = \det A = d!\Delta(u_0, \cdots, u_d)$.

□

Let us go back to the proof of Theorem 3. Applying the preceding change of variables to the calculation of the integral in the equality (10), we obtain

$$\mathbf{E}\{h(\mathcal{S})\mathbf{1}_{\{\widehat{\Phi}\cap A=\emptyset\}}\} = \frac{2^d}{\omega_d {\omega_{d-1}}^d V_d(B)(d+1)} \int \mathcal{I}(u)\Delta(u)\mathbf{1}_A(u) d\overline{\nu_d}^{(d+1)}(u), \tag{13}$$

where

$$\mathcal{I}(u) = \int_0^{+\infty} h(S(Ru_0, \cdots, Ru_d)) \int_B \mathbf{P}\left\{H(x)\cap(z+B(R)) = \emptyset; \left(x - \frac{x}{||x||^2}(x\cdot z)\right) \not\in A \, \forall x \in \Phi\right\} dz dR.$$

Let us notice that

$$\left\{(H(x)-z)\cap B(R) = \emptyset; \left(x - \frac{x}{||x||^2}(x\cdot z)\right) \not\in A \, \forall x \in \Phi\right\}$$
$$= \left\{H\left(x - \frac{x}{||x||^2}(x\cdot z)\right)\cap B(R) = \emptyset; \left(x - \frac{x}{||x||^2}(x\cdot z)\right) \not\in A \, \forall x \in \Phi\right\} \tag{14}$$

Besides, the set $\left\{x - \frac{x}{||x||^2}(x\cdot z); x\in\Phi\right\}$ is distributed as Φ (see for example [5]). Consequently, we deduce from (14) and the Poissonian property of Φ that for all $z \in \mathbb{R}^d$,

$$\mathbf{P}\left\{(H(x)-z)\cap B(R) = \emptyset; \left(x - \frac{x}{||x||^2}(x\cdot z)\right) \not\in A \, \forall x \in \Phi\right\}$$
$$= \mathbf{P}\{\Phi\cap(B(R)\cup A) = \emptyset\} = \exp\{-\mu(A\setminus B(R))\}\exp\{-\sigma_d R\} \tag{15}$$

Inserting (15) in the equation (13), we get that:
_ the inball radius of $\mathcal{S}$ is exponentially distributed with parameter σ_d.
_ the directions from the origin to the points of tangency of $\mathcal{S}$ with its inball have a joint distribution given by (11) and are independent from the inball radius.
_ Conditionally to the fact that the inball radius of $\mathcal{S}$ is equal to r, $r > 0$, $\widehat{\Phi}$ is distributed as a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c} d\mu$.

So the property (9) provides us the required construction of $\mathcal{C}$.

□

3 Some consequences of the method concerning geometric characteristics of the Poissonian tessellation.

In this section, we show how to use the same method to obtain new (or not) informations about the typical cell.

As an example, let us denote by $N_k(\mathcal{C})$ (resp. $V_k(\mathcal{C})$) the number of k-dimensional faces (resp. the k-dimensional Hausdorff measure) of the typical cell $\mathcal{C}$, $0 \leq k \leq d$.

Theorem 4 *We have:*
(1) $\mathbf{E}V_k(\mathcal{C}) = \frac{2^d\binom{d}{k}}{{\omega_{d-1}}^k\omega_k}$,
(2) $\mathbf{E}N_k(\mathcal{C}) = 2^{d-k}\binom{d}{k}$
(3) $\mathbf{P}\{N_{d-1}(\mathcal{C}) = d+1\} = \frac{2^{d+1}}{d(d+1)\omega_d^2{\omega_{d-1}}^d}\int \frac{\Delta(u)}{b(u)}\mathbf{1}_A(u)d\overline{\nu_d}^{d+1}(u)$,
where $b(u)$, $u \in A$, denotes the mean width of the simplex which admits u as the set of its contact points with its inball.

Proof. **(1)** Let us first recall a well-known result due to R. E. Miles [12]:

Lemma 2 (Miles, 1969) *The intersection of a Poisson hyperplane process of $\mathbb{R}^d$ of intensity measure given by (2) with a affine sub-space of $\mathbb{R}^d$ of dimension k, $0 \leq k \leq d$, is equal in law to a Poisson hyperplane process of $\mathbb{R}^k$ of intensity measure $\mu_{k,d}$ given by:*

$$\mu_{k,d}(A) = \frac{\omega_{d-1}}{\omega_{k-1}}\int_0^{+\infty}\int_{\mathbb{S}^{k-1}} \mathbf{1}_A(r,u)d\nu_k(u)dr, \quad A \in \mathcal{B}(\mathbb{R}^k). \tag{16}$$

Let us show the following intermediary lemma:

Lemma 3 *Let us consider for all $0 \leq k \leq d$, the measure Λ_k defined by*

$$\Lambda_k(B) = \mathbf{E}\sum_{F\in\mathcal{F}_k} V_k(B\cap F), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where $\mathcal{F}_k$ denotes the set of all k-dimensional faces of the tessellation and V_k is the k-dimensional Hausdorff measure.
Then the equality

$$\Lambda_k = c_{k,d}\cdot V_d \tag{17}$$

is satisfied, $c_{k,d}$ being the mean k-dimensional measure of the tessellation per unit of volume such that

$$c_{k,d} = \binom{d}{k}\frac{\omega_d{\omega_{d-1}}^{d-k}}{\omega_k 2^{d-k}}. \tag{18}$$

Proof of Lemma 3. The result (17) comes from the fact that the measure Λ_k is invariant by any translation in $\mathbb{R}^d$, so is proportional to the Lebesgue measure of $\mathbb{R}^d$.

Let us show the result (18) by a reasoning of induction: it is clearly verified for $k = d$. Let $0 \leq k \leq d-1$. Taking B equal to the unit-ball of $\mathbb{R}^d$, we have

$$\begin{aligned}\Lambda_k(B) &= c_{k,d}\omega_d \\ &= \mathbf{E}\sum_{\{x_1,\cdots,x_{d-k}\}\in\Phi} V_k(B\cap H(x_1)\cap\cdots\cap H(x_{d-k})).\end{aligned}$$

Applying Slivnyak's formula (1), we obtain

$$\begin{aligned}
&\Lambda_k(B)\\
&= \frac{1}{(d-k)!}\int V_k(B\cap H(x_1)\cap\cdots\cap H(x_{d-k}))d\overline{\mu}^{(d-k)}(x) \qquad (19)\\
&= \frac{1}{(d-k)!}\int\int V_k\{[B\cap H(x_1)\cap\cdots\cap H(x_{d-1-k})]\cap H(x_{d-k})\}d\mu(x_{d-k})d\overline{\mu}^{(d-1-k)}(x).
\end{aligned}$$

Applying Lemma 2 to the section of the tessellation with the $(k+1)$-dimensional space $H(x_1)\cap\cdots H(x_{d-k-1})$, we obtain:

$$\Lambda_k(B) = \frac{1}{(d-k)}\omega_d c_{k+1,d}\frac{\omega_{d-1}}{\omega_k}c_{k,k+1}. \qquad (20)$$

Besides, applying (19) to $k=d-1$, we get that

$$\begin{aligned}
c_{d-1,d} &= \frac{1}{\omega_d}\int_B V_{d-1}(B\cap H(tu))dtd\nu_d(u)\\
&= \frac{\omega_{d-1}}{\omega_d}\sigma_d\int_0^1(1-t^2)^{\frac{d-1}{2}}dt = \frac{\sigma_d}{2}.
\end{aligned}$$

Consequently, inserting the equality $c_{k,k+1} = \sigma_{k+1}/2$ in (20), we deduce the following relation of induction:

$$c_{k,d} = \frac{k+1}{2(d-k)}\omega_{d-1}\frac{\omega_{k+1}}{\omega_k}c_{k+1,d},$$

which, after iteration, gives us the formula

$$c_{k,d} = \binom{d}{k}\frac{\omega_d{\omega_{d-1}}^{d-k}}{\omega_k 2^{d-k}}.$$

□

Let us go back to the determination of $\mathbf{E}V_k(\mathcal{C})$, $0\le k\le d$: repeating an argument of Møller (see [16], page 62), we consider for all convex polyhedron P, $\mathcal{E}_k(P)$ the set of all k-dimensional faces of P. If $B\subset\mathbb{R}^d$ is a fixed Borel set such that $0<V_d(B)<+\infty$, we then have

$$\begin{aligned}
V_d(B)c_{k,d} &= \mathbf{E}\sum_{F\in\mathcal{F}_k}V_k(B\cap F)\\
&= \frac{1}{2^{d-k}}\mathbf{E}\sum_{z\in\Psi}\sum_{F\in\mathcal{E}_k(C(z))}V_k(B\cap F),
\end{aligned}$$

the last equality being due to the fact that any k-dimensional face of the tessellation is in the boundary of exactly 2^{d-k} different cells.

Using formula (5) we obtain that

$$\begin{aligned}
V_d(B)c_{k,d} &= \frac{\omega_d {\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}}\int \mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C}+x)} V_k(B\cap F)dx \\
&= \frac{\omega_d {\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}}\int \mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C})} V_k(B\cap (F+x))dx \\
&= \frac{\omega_d {\omega_{d-1}}^d}{2^d}\frac{1}{2^{d-k}} V_d(B)\mathbf{E}\sum_{F\in\mathcal{E}_k(\mathcal{C})} V_k(F),
\end{aligned}$$

the last equality being deduced from Fubini's theorem. Consequently, we deduce that

$$\begin{aligned}
\mathbf{E}V_k(\mathcal{C}) &= 2^{d-k}c_{k,d}\frac{2^d}{\omega_d {\omega_{d-1}}^d} \\
&= \frac{2^d\binom{d}{k}}{{\omega_{d-1}}^k\omega_k}.
\end{aligned}$$

(2) We consider the proces Ψ_k, $0\le k\le d$, of the centers of the inballs of the k-dimensional faces of the tessellation. Ψ_k is stationary and we will note λ_k its intensity. Besides, we define for any $z\in\Psi_k$, $F(z)$ as the unique k-dimensional face associated to z.
Then the typical k-dimensional face $\mathcal{C}_{k,d}$ associated to the tessellation is well-defined by the following formula:

$$\mathbf{E}h(\mathcal{C}_{k,d}) = \frac{1}{V_d(B)\lambda_k}\mathbf{E}\sum_{z\in\Psi_k\cap B} h(F(z)-z),$$

for all bounded measurable function h and any fixed Borel set $B\subset\mathbb{R}^d$, satisfying $0<V_d(B)<+\infty$.
Let us notice in particular that the typical d-face $\mathcal{C}_{d,d,}$ is the classical typical cell $\mathcal{C}$ associated to the tessellation. The following lemma gives a characterization of the law of the typical k-face and can be easily deduced from a joint use of Slivnyak's formula and Lemma 2:

Lemma 4 *$\mathcal{C}_{k,d}$ is equal in law to the typical cell of a Poissonian tessellation in $\mathbb{R}^k$ with intensity measure given by the formula (16).*

A direct consequence of the preceding lemma and the point (1) of Theorem 4 is that

$$\begin{aligned}
\mathbf{E}V_k(\mathcal{C}_{k,d}) &= \frac{2^k}{\omega_k {\omega_{k-1}}^k}\times\left(\frac{\omega_{k-1}}{\omega_{d-1}}\right)^k \\
&= \frac{2^k}{\omega_k {\omega_{d-1}}^k}. \qquad (21)
\end{aligned}$$

Following the same method as Møller [16] (prop. 3.2.2) for Voronoi tessellations, we can show that:

$$\mathbf{E}V_k(\mathcal{C}_{k,d}) = \mathbf{E}V_k(\mathcal{C})/\mathbf{E}N_k(\mathcal{C}). \qquad (22)$$

Consequently, from (21) and (22), we obtain:

$$\mathbf{E}N_k(\mathcal{C}) = 2^{d-k}\binom{d}{k}.$$

(3) Considering the construction of $\mathcal{C}$ obtained in Theorem 3, and denoting by Φ_r, $r>0$, a Poisson point process of intensity measure $\mathbf{1}_{B(r)^c}d\mu$, we have

$$\mathbf{P}\{N_{d-1}(\mathcal{C}) = d+1\} \\ = \frac{2^d}{(d+1)\omega_d{\omega_{d-1}}^d}\int \mathbf{P}\{\#(\Phi_r\cap S(Ru))=0\}e^{-\sigma_d R}\Delta(u)\mathbf{1}_A(u)dRd\overline{\nu_d}^{(d+1)}(u).$$

Let us then notice that it is well-known (see for example [4]) that

$$\mathbf{P}\{\#(\Phi_r\cap S(Ru))=0\} = \exp\left\{-R\left(\frac{\sigma_d}{2}b(u)-\sigma_d\right)\right\},$$

where $b(u)$ denotes the mean width of the simplex $S(u)$. Consequently, after an integration with respect to R, we deduce

$$\mathbf{P}\{N_{d-1}(\mathcal{C}) = d+1\} = \frac{2^{d+1}}{d(d+1)\omega_d^2{\omega_{d-1}}^d}\int \frac{\Delta(u)}{b(u)}\mathbf{1}_A(u)d\overline{\nu_d}^{d+1}(u).$$

□

Remark 1 Let us remark that it is possible to use the point (1) of Theorem 4 to obtain the expression of the intensity λ_d of the process Ψ of the centers of the inballs of the cells, and more precisely to show that λ_d has the same value as the intensity $c_{0,d}$ of the process of the vertices of the tessellation: actually, we could define the typical cell by associating to each cell constituting the tessellation its lowest vertex (which exists a.s.) and by replacing the process Ψ of the centers of the inballs by the process Λ of the lowest vertices. It is easy to notice that any vertex of the tessellation is the lowest vertex of exactly one cell, which means that Λ can be exactly identified to the process of the vertices of the tessellation, of intensity $c_{0,d}$. We can also remark that the two definitions of the typical cell provide the same law because in the two cases, the equality (6) is satisfied.

Repeating the argument of Møller [16] (page 62), we obtain that:

$$\mathbf{E}V_d(\mathcal{C}) = {\lambda_d}^{-1} = {c_{0,d}}^{-1}.$$

Remark 2 The formula (1) of Theorem 4 was previously obtained by Matheron [9]. The equalities (2) and (3) were given by R. E. Miles [12] when $d=2,3$; for any $d>3$, they are new.

Aknowledgement. I thank Professor A. Goldman for having suggested me the idea of using Slivnyak's formula in order to generalize the theorem of Miles to any dimension.

References

[1] P. Calka. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6):557–562, 2001.

[2] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *To appear in Adv. in Appl. Probab.*, 2002.

[3] R. Cowan. Properties of ergodic random mosaic processes. *Math. Nachr.*, 97:89–102, 1980.

[4] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[5] A. Goldman. Sur une conjecture de D. G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne. *Ann. Probab.*, 26(4):1727–1750, 1998.

[6] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[7] S. Goudsmit. Random distribution of lines in a plane. *Rev. Modern Phys.*, 17:321–322, 1945.

[8] I. N. Kovalenko. A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons. *J. Appl. Math. Stochastic Anal.*, 12(4):301–310, 1999.

[9] G. Matheron. *Random sets and integral geometry.* John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[10] R. E. Miles. Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci. U.S.A.*, 52:901–907, 1964.

[11] R. E. Miles. Random polygons determined by random lines in a plane. II. *Proc. Nat. Acad. Sci. U.S.A.*, 52:1157–1160, 1964.

[12] R. E. Miles. Poisson flats in Euclidean spaces. I. A finite number of random uniform flats. *Advances in Appl. Probability*, 1:211–237, 1969.

[13] R. E. Miles. The various aggregates of random polygons determined by random lines in a plane. *Advances in Math.*, 10:256–290, 1973.

[14] J. Møller. A simple derivation of a formula of Blaschke and Petkantschin. *Research report, Dept. of Theor. Stat., University of Aarhus*, 52, 1985.

[15] J. Møller. Random tessellations in $\mathbb{R}^d$. *Adv. in Appl. Probab.*, 21(1):37–73, 1989.

[16] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[17] K. Paroux. Quelques théorèmes centraux limites pour les processus Poissoniens de droites dans le plan. *Adv. in Appl. Probab.*, 30(3):640–656, 1998.

Chapitre 5

Une preuve rigoureuse d'un résultat de R. E. Miles sur les mosaïque poissoniennes épaissies.

5.1 Introduction.

Dans cette partie, Φ désignera un processus ponctuel de Poisson sur $\mathbb{R}^d$, $d \geq 2$, de mesure d'intensité μ (définie par (1.4)), $\mathcal{H}$ l'ensemble des hyperplans polaires associés à Φ (voir (1.5)).

Rappelons que les mosaïques poissoniennes de droites dans le plan modélisent en particulier la répartition des fibres d'une feuille de papier. En 1964, prenant en compte le fait que dans la réalité les fibres ont une épaisseur non nulle, R. E. Miles s'intéresse aux mosaïques poissoniennes épaissies. Cela lui donne par ailleurs un moyen de montrer que le rayon du disque inscrit de la cellule typique d'une mosaïque poissonienne suit une loi exponentielle. Plus précisément, on épaissit chaque hyperplan de $\mathcal{H}$ d'une épaisseur fixée $e \geq 0$ et l'on s'intéresse à l'ensemble des composantes connexes du complémentaire dans l'espace euclidien $\mathbb{R}^d$ de ces hyperplans épaissis. Miles affirme que les moyennes empiriques limites sur ces nouvelles cellules qui sont aussi des polyèdres, existent et sont les mêmes que dans le cas d'une mosaïque poissonienne classique.

Sa démarche originale est néanmoins heuristique et insatisfaisante au niveau mathématique (de notre époque...). En effet, la convergence des moyennes ergodiques des mosaïques épaissies n'a pas été démontrée et il n'est pas clair que la limite obtenue soit indépendante de l'épaisseur choisie.

Nous prouvons cette convergence à épaisseur fixée en suivant une technique analogue à celle du paragraphe 1.5.1. Le fait que la limite obtenue soit indépendante de l'épaisseur provient directement de l'invariance de la mesure μ par translation radiale. En d'autres termes, pour tout $r \geq 0$, l'application

$$\begin{cases} \mathbb{R}^d \setminus \{0\} & \longrightarrow & \mathbb{R}^d \\ x & \longmapsto & (||x|| + r) \cdot \frac{x}{||x||}, \end{cases}$$

envoie la mesure μ sur la mesure $\mathbf{1}_{B(0,r)^c} d\mu$.

Nous concluons ce travail en précisant l'idée de Miles permettant de déterminer la loi empirique du rayon de la boule inscrite à partir de la propriété évoquée ci-dessus.

A rigorous proof of a result of R. E. Miles concerning the thickened Poisson hyperplane process in $\mathbb{R}^d$. *

Pierre Calka†

Abstract

R. E. Miles introduced in [12] the notion of empirical distribution and associated typical cell for a Poisson line process in the plane. In [10], he gave to each line of the process a fixed width and claimed that the empirical means of the polygon aggregate comprising the interstices between these thick lines were independent from the thickness. Nevertheless, his results were based on certain heuristic considerations: on one hand, he did not consider the "edge effects" for showing the convergence of empirical means, on the other hand he did not formalize the definition of empirical cell in the thick-lines case.

In this paper, we give detailed proofs for a Poissonian tessellation in any dimension of the two following points: the insignificance of "edge regions", which provides the existence of the empirical distributions, and the equality in law of the typical cells in the classical and in the thickened Poissonian tessellations.

Introduction.

Let Φ be a Poisson point process in $\mathbb{R}^d$, $d \geq 2$, of intensity measure

$$\mu(A) = \int_0^{+\infty} \int_{\mathbb{S}^{d-1}} \mathbf{1}_A(r,u) d\nu_d(u) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where ν_d is the area measure of the unit-sphere $\mathbb{S}^{d-1}$.

Let us consider for all $x \in \mathbb{R}$, $H(x) = \{y \in \mathbb{R}^d; (y-x)\cdot x = 0\}$, ($x \cdot y$ being the usual scalar product). Then the set $\mathcal{H} = \{H(x); x \in \Phi\}$ divides the space into convex polyhedra that constitute the so-called *d-dimensional Poissonian tessellation.* This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.

* *American Mathematical Society 2000 subject classifications.* Primary 60D05; secondary 60G55.
Key words and phrases. Empirical distributions, stochastic geometry, thickened Poisson hyperplane process, typical cell.
Running head. A rigorous proof of a result of Miles about thickened Poisson hyperplane process.

† *Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr, *Phone number*: +33 (0) 4 37 28 74 47

This random object was used for the first time by S. A. Goudsmit [8] and by R. E. Miles ([10], [11] and [12]). In particular, it provides a model for the fibrous structure of sheets of paper.

Miles introduced in particular the notion of *empirical* (or *typical*) cell associated to the tessellation. Recent contributions about the law of the area of the typical cell and the famous D. G. Kendall conjecture were provided by A. Goldman [6] and I. N. Kovalenko [9]. The fundamental frequencies of the cells have been studied by A. Goldman [5] and have been useful to obtain informations about the frequencies of the cells of the Poisson-Voronoi tessellation [7]. Central limit theorems have been provided by K. Paroux [14] in this context. Besides, we have obtained a new interpretation of the typical cell by means of a Palm measure [1], and the explicit distribution of the radius of the smallest disk centered at the origin containing the Crofton cell in the plane [2].

If we give to each hyperplane of $\mathcal{H}$ a fixed thickness e, the connex components of the remaining parts of $\mathbb{R}^d$ constitute a new set of cells. Since in reality fibers of sheets of paper always have a width, this thickened Poisson line process provides a more accurate model. Besides, Miles used it to determine the empirical law of the indisk radius. He claimed in [10] that for the two-dimensional case the associated empirical distributions do not depend on the thickness e. The goal of this paper is to prove this fact precisely in any dimension.

In the first section, we give a rigorous proof of the convergence of the empirical means of a classical Poissonian tessellation of $\mathbb{R}^d$, taking the "edge effects" into account (for a proof of a more precise fact in dimension two using an other method, see papers of Paroux [13], [14]). We here follow essentially the method described by R. Cowan in [3] and [4] in dimension two. The treatment of the d-dimensional case, $d > 2$, raises some new difficulties of geometrical nature (see in particular Lemma 3).

We then consider in the second section the thickened Poissonian tessellation introduced by Miles [10] and we use the tools of the preceding section to show that the law of the empirical cell is independent of the thickness. We conclude with an easy consequence of the thickening, following an idea given by Miles [10] for the dimension two: the empirical law of the inball radius is exponential. Let us notice that this result was a part of a more general theorem in the note [1], obtained by completely different methods.

1 The empirical distributions and the typical cell of the Poissonian tessellation.

In this section, we recall how to define the empirical distributions of the d-dimensional Poissonian tessellation. The main result , i.e. the treatment of the edge effects, is contained in Proposition 1.

Let Φ be a Poisson point process in $\mathbb{R}^d$ of intensity measure

$$\mu(A) = \mathbf{E}\sum_{x\in\Phi}\mathbf{1}_A(x) = \int_0^{+\infty}\int_{\mathbb{S}^{d-1}}\mathbf{1}_A(r,u)d\nu_d(u)dr, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

For all $x \in \mathbb{R}^d$, let us define the polar hyperplane at x

$$H(x) = \{y \in \mathbb{R}^d; (y - x) \cdot x = 0\},$$

where $\cdot$ is the usual inner product in $\mathbb{R}^d$.
We call the set

$$\mathcal{H}(\Phi) = \{H(x_i); x_i \in \Phi\}$$

a *Poisson hyperplane process* in $\mathbb{R}^d$ and the closure of a connex component of the set

$$\mathbb{R}^d \setminus \cup_{x_i \in \Phi} H(x_i)$$

a *cell* associated to $\mathcal{H}$. Then almost surely, the cells are bounded convex polytopes which constitute a tessellation of $\mathbb{R}^d$, called a *Poissonian tessellation*.
Moreover , if $x \in \mathbb{R}^d$ is fixed, then almost surely no hyperplane of $\mathcal{H}$ contains x, so we will denote by $C_x(\Phi)$ the cell containing the point x. In particular, we call $C_0(\Phi)$ the *Crofton cell*.

We realize Ω as the space $\mathcal{M}_\sigma$ of the locally finite sets of $\mathbb{R}^d$ endowed with the cylindric σ-field

$\mathcal{T}_c$ generated by the applications

$$\varphi_A : \begin{cases} \mathcal{M}_\sigma & \longrightarrow & \mathbb{N} \cup \{+\infty\} \\ \gamma & \longmapsto & \#(A \cap \gamma) \end{cases} \quad , \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Φ then is the identity application on Ω.

For all $a \in \mathbb{R}^d$, let us remark that

$$a + H(x_i) = H(t^a(x_i)) \quad \text{with } t^a(x_i) = \left(1 + \frac{x_i \cdot a}{||x_i||^2}\right) x_i, i \geq 1.$$

The correspondence

$$\{x_i; i \geq 1\} \longmapsto \{t^a(x_i); i \geq 1\}$$

induces classically a transformation $T^a : \Omega \longrightarrow \Omega$ preserving the measure [6].

It is well-known that:

Lemma 1 *For all $a \in \mathbb{R}^d$, the transformations T^a are ergodic.*

Proof. It suffices to show that the measure is strongly mixing, which means that for any bounded $A, B \in \mathcal{B}(\mathbb{R}^d)$, $k, l \in \mathbb{N}$, we have when $n \to +\infty$,

$$\begin{aligned} &\mathbf{P}[\{\#(T^{-na}(\Phi) \cap A) = k\} \cap \{\#(\Phi \cap B) = l\}] \\ &\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\}] \\ &\qquad \to \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\}. \end{aligned} \tag{1}$$

Let $D_\alpha = \{x \in \mathbb{R}^d; |x \cdot a| \geq \alpha\}$, $\alpha > 0$.
_ Supposing there exists $\alpha > 0$ such that $B \subset D_\alpha$, then for n large enough, $t^{na}(A) \cap B = \emptyset$, and so we have by the Poissonian property of Φ,

$$\begin{aligned} &\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\}] \\ &\qquad = \mathbf{P}\{\#(\Phi \cap t^{na}(A)) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\} \\ &\qquad = \mathbf{P}\{\#(T^{-na}(\Phi) \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\} \\ &\qquad = \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\}. \end{aligned}$$

- In the general case, we consider $\varepsilon \in (0,1)$ and take $\alpha > 0$ such that the event $E_\alpha = \{\Phi \cap D_\alpha^c \cap B = \emptyset\}$ satisfies

$$P(E_\alpha) \geq 1 - \varepsilon. \tag{2}$$

Applying the first case to $B \cap D_\alpha$, we then have for n large enough,

$$\begin{aligned}
&\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\} \cap E_\alpha] \\
&\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B \cap D_\alpha) = l\} \cap E_\alpha] \\
&\qquad = \mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] \cdot \mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\}. \qquad (3)
\end{aligned}$$

Moreover, using (2), we have

$$\begin{aligned}
&|\mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\} - \mathbf{P}\{\#(\Phi \cap B) = l\}| \\
&\qquad \leq |\mathbf{P}\{\#(\Phi \cap B \cap D_\alpha) = l\} - \mathbf{P}[\{\#(\Phi \cap B \cap D_\alpha) = l\} \cap E_\alpha]| \\
&\qquad\qquad + |\mathbf{P}[\{\#(\Phi \cap B) = l\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap B) = l\}| \\
&\qquad \leq 2\varepsilon. \qquad (4)
\end{aligned}$$

Besides,

$$\begin{aligned}
&|\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap A) = k\}| \\
&\qquad = |\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap E_\alpha] - \mathbf{P}\{\#(\Phi \cap t^{na}(A)) = k\}| \leq \varepsilon. \qquad (5)
\end{aligned}$$

Consequently, for n large enough, using (3), (4) and (5), we get that

$$|\mathbf{P}[\{\#(\Phi \cap t^{na}(A)) = k\} \cap \{\#(\Phi \cap B) = l\}] - \mathbf{P}\{\#(\Phi \cap A) = k\} \cdot \mathbf{P}\{\#(\Phi \cap B) = l\}| \leq 4\varepsilon,$$

which proves the convergence (1).

□

Let us denote by $\mathcal{C}_R$ (resp. $\mathcal{C}'_R$) the set of cells included in the open ball $B(R)$ centered at the origin, with radius $R > 0$ (resp. intersecting the boundary of $B(R)$) and $N_R = \#\mathcal{C}_R$ (resp. $N'_R = \#\mathcal{C}'_R$).

Let us prove the integrability of $(N_R + N'_R)$. The set of the hyperplanes of $\mathcal{H}(\Phi)$ intersecting $B(R)$ divides the space into $2^{\#(\Phi \cap B(R))}$ connex components. Consequently, since $\#(\Phi \cap B(R))$ is a Poisson variable,

$$\mathbf{E}(N_R + N'_R) \leq \mathbf{E}2^{\#(\Phi \cap B(R))} < +\infty.$$

Besides, we consider the space $\mathcal{K}$ of the convex compact sets of $\mathbb{R}^d$ endowed with the usual Hausdorff topology and $h : \mathcal{K} \longrightarrow \mathbb{R}_+$ a translation-invariant, positive and bounded measurable function. We will note $K_h > 0$ a bound for h.

By applying the equalities $C_0(T^{-a}\omega) = C_a(\omega) - a$, $a \in \mathbb{R}^d$, $\omega \in \Omega$, we get the identities:

$$\frac{1}{v(R)} \int_{B(R)} \frac{dx}{V_d(C_0(T^{-x}\omega))} = \frac{N_R(\omega)}{v(R)} + \frac{1}{v(R)} \varepsilon(R, \mathbf{1}, \omega) \tag{6}$$

$$\frac{1}{v(R)} \int_{B(R)} \frac{h(C_0(T^{-x}\omega))}{V_d(C_0(T^{-x}\omega))} dx = \frac{1}{v(R)} \sum_{C \in \mathcal{C}_R} h(C_i(\omega)) + \frac{1}{v(R)} \varepsilon(R, h, \omega), \tag{7}$$

where V_d is the d-dimensional Lebesgue measure of $\mathbb{R}^d$, $v(R) = V_d(B(R))$, and

$$\varepsilon(R, h, \cdot) \quad = \sum_{C \in \mathcal{C}'_R} h(C) \cdot \frac{V_d(C \cap B(R))}{V_d(C)} \text{ a.s..}$$

Taking the expectation of (7), we get that

$$\mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right) \leq \frac{K_h}{v(R)} \mathbf{E}(N_R + N'_R) < +\infty.$$

Because of ergodicity of the transformations T^a, $a \in \mathbb{R}^d$, we can apply Wiener's ergodic theorem [15] and so, assuming that when $R \to +\infty$, the contribution of the rest $\varepsilon(R, h, \omega)$ disappears, i.e.

$$\varepsilon(R, h, \cdot)/v(R) \to 0 \text{ a.s. when } R \to +\infty, \tag{8}$$

then the empirical mean of h, that means the almost sure limit

$$\widetilde{\mathbf{E}}h = \lim_{R \longrightarrow \infty} \frac{1}{N_R} \sum_{C \in \mathcal{C}_R} h(C) \tag{9}$$

is well defined and coincides with $\{\mathbf{E}\,(1/V_d(C_0))\}^{-1}\,\mathbf{E}\,(h(C_0)/V_d(C_0))$.

It remains to see this process is well justified, i.e. the convergence (8) is true. We have a.s. the inequality

$$\frac{\varepsilon(R, h, \cdot)}{v(R)} \leq K_h \cdot \frac{N'_R}{v(R)}.$$

As a consequence, it suffices to show the following proposition, which is the main difficulty in the construction of the empirical distributions:

Proposition 1 *When $R \to +\infty$, we have:*

$$N'_R/v(R) \to 0 \quad a.s..$$

Proof. We propose here a generalization to any dimension of the argument exposed by R. Cowan [3], [4] in the two-dimensional case. In order to prove Proposition 1, we have to introduce some new notations. Let us consider for every $0 \leq k \leq d$:

(i) $N_{R,k}$ the number of faces of dimension k of the tessellation included in $B(R)$;

(ii) $N'_{R,k}$ the number of faces of dimension k intersecting the boundary of $B(R)$;

(iii) $S_{R,k}$ the k-dimensional Hausdorff measure of the intersection of $B(R)$ with the k-dimensional faces of the tessellation.

In particular, let us notice that $N_R = N_{R,d}$ (resp. $N'_R = N'_{R,d}$). Let us prove that all these random variables are integrable. The set of the hyperplanes of $\mathcal{H}(\Phi) \cap B(R)$ induces at most $\left(2^{\#(\Phi\cap B(R))}\binom{\#(\Phi\cap B(R))}{d-k}\right)$ k-faces, $0\le k\le d$, in the tessellation. Consequently, since $\#(\Phi\cap B(R))$ is a Poisson variable,

$$\mathbf{E}(N_{R,k}+N'_{R,k})\le \mathbf{E}\left(2^{\#(\Phi\cap B(R))}\binom{\#(\Phi\cap B(R))}{d-k}\right)<+\infty.$$

Moreover, we have clearly

$$S_{R,k}\le \frac{\sigma_k}{k}R^k(N_{R,k}+N'_{R,k}) \text{ a.s.},$$

so $S_{R,k}$ is also integrable for all $0\le k\le d$.

To prove Proposition 1, we need two intermediate lemmas:

Lemma 2 *For every $0\le k\le d$, $S_{R,k}/v(R)$ converges a.s. to a constant when R goes to infinity.*

Lemma 3 *For every $0\le k\le d$, $N'_{R,k}/v(R)$ converges to zero a.s. when R goes to infinity.*

Applying Lemma 3 to $k=d$, we clearly obtain Proposition 1.

□

Let us now focus on the proof of the two lemmas:

Proof of Lemma 2. Let us fix $0\le k\le d$ and show the following intermediary result:

$$\int_{B(R-y)} S_{y,k}(T^{-t}(\omega))dt \le v(y)S_{R,k}(\omega)\le \int_{B(R+y)} S_{k,y}(T^{-t}(\omega))dt \text{ a.s., } 0<y<R. \quad (10)$$

Denoting by $\mathcal{F}_{k,R}$ the set of k-faces of the tessellation intercepting $B(R)$ and by $\mu_{k,F}$ the k-dimensional Hausdorff measure of the face F, $F\in\mathcal{F}_{k,R}$, we actually have

$$\begin{aligned}
\int_{B(R-y)} S_{y,k}(T^{-t}(\omega))dt &= \int \mathbf{1}_{B(R-y)}(t)\sum_{F\in\mathcal{F}_{k,R}(\omega)}\int \mathbf{1}_{B(y)}(s)\mathbf{1}_F(s+t)d\mu_{k,F}(s+t)dt\\
&= \sum_{F\in\mathcal{F}_{k,R}(\omega)}\int \mathbf{1}_{B(R-y)}(t)\int \mathbf{1}_{B(y)}(u-t)\mathbf{1}_F(u)d\mu_{k,F}(u)dt\\
&\le \sum_{F\in\mathcal{F}_{k,R}(\omega)}\int\left[\int \mathbf{1}_{B(y)}(u-t)dt\right]\mathbf{1}_{B(R)}(u)\mathbf{1}_F(u)d\mu_{k,F}(u)\\
&\le v(y)\cdot S_{R,k}(\omega) \quad \text{a.s.,}
\end{aligned}$$

which proves the right part of the estimation (10).

Moreover, we have also

$$\begin{aligned}
\int_{B(R+y)} S_{y,k}(T^{-t}(\omega))dt &= \sum_{F\in\mathcal{F}_{k,R+2y}(\omega)}\int\left[\int \mathbf{1}_{B(R+y)}(t)\mathbf{1}_{B(y)}(u-t)dt\right]\mathbf{1}_F(u)d\mu_{k,F}(u)\\
&\ge \sum_{F\in\mathcal{F}_{k,R}(\omega)}\int\left[\int \mathbf{1}_{B(y)}(u-t)dt\right]\mathbf{1}_{B(R)}(u)\mathbf{1}_F(u)d\mu_{k,F}(u)\\
&\ge v(y)\cdot S_{R,k}(\omega) \quad \text{a.s.,}
\end{aligned}$$

which completes the proof of the result (10).

Going back to the proof of Lemma 2, (10) implies that a.s. we have

$$\frac{v(R-y)}{v(y)v(R)}\int_{B(R-y)}\frac{S_{y,k}(T^{-t}(\omega))}{v(R-y)}dt \leq \frac{S_{R,k}(\omega)}{v(R)} \leq \frac{v(R+y)}{v(y)v(R)}\int_{B(R+y)}\frac{S_{y,k}(T^{-t}(\omega))}{v(R+y)}dt.$$

Using Wiener's ergodic theorem, the left and right expressions tend to $(\mathbf{E}(S_{y,k})/v(y))$ a.s.. So the proof of the a.s. convergence of $S_{R,k}/v(R)$ is completed.

$\square$

Proof of Lemma 3. We prove the convergence

$$(C)_k\colon \quad N'_{R,k}/v(R) \to 0 \text{ a.s. when } R \to +\infty,$$

by a reasoning of induction on $k \in [0, d]$.

Since $N'_{R,0} = 0$ a.s., $(C)_0$ is true. Let us suppose $(C)_{k-1}$ is satisfied for a fixed $1 \leq k \leq d-1$ and show $(C)_k$.

Let us consider a fixed $y \in (0, R)$. If a k-face intersects the boundary of $B(R)$ in such a way that the intersection of its sub-$(k-1)$-faces with the set $B(R+y)\setminus B(R)$ is empty, then the k-dimensional measure of the intersection of this k-face with $B(R+y)\setminus B(R)$ is at least equal to the volume of a k-dimensional ball of radius $\sqrt{(R+y)^2-R^2}$, i.e.

$$A_k = \omega_k((R+y)^2 - R^2)^{k/2},$$

where ω_k is the k-dimensional Lebesgue measure of the unit-ball in $\mathbb{R}^k$.

Consequently, denoting by L_k the number of such k-faces, we have

$$S_{R+y,k} - S_{R,k} \geq A_k \cdot L_k, \tag{11}$$

Moreover, we easily get the formula

$$\begin{aligned} L_k \quad = \quad & N'_{R,k} - \#\{k\text{-faces having one of its } (k-1)\text{-faces} \\ & \qquad\qquad \text{which intersects } B(R+y)\setminus B(R)\}. \end{aligned} \tag{12}$$

Besides, for every $0 \leq l \leq l' \leq d$, a fixed l-dimensional face of the Poissonian tessellation is included in exactly $2^{l'-l}\binom{d-l}{d-l'}$ different l'-dimensional faces.

In particular, a fixed $(k-1)$ face is included in exactly $2(d-k+1)$ k-faces. A direct consequence of this fact and (12) is that

$$L_k \quad \geq N'_{R,k} - 2(d-k+1)\#\{(k-1)\text{-faces intersecting } B(R+y)\setminus B(R)\}. \tag{13}$$

It remains to see that a $(k-1)$-face intersecting $B(R+y)\setminus B(R)$ may:
- either intersect the boundary of $B(R+y)$;
- either intersect the boundary of $B(R)$;
- or be included in $B(R+y)\setminus B(R)$.

In the last case, we notice that the $(k-1)$-face has a 0-face included in $B(R+y)\setminus B(R)$. So we obtain that

$$\begin{aligned}
&\#\{(k-1)\text{-faces intersecting } B(R+y)\setminus B(R)\}\\
&\le N'_{R+y,k-1}+N'_{R,k-1}+\#\{(k-1)\text{-faces included in } B(R+y)\setminus B(R)\}\\
&\le N'_{R+y,k-1}+N'_{R,k-1}+2^{k-1}\binom{d}{k-1}\cdot\#\{0\text{-faces included in } B(R+y)\setminus B(R)\}\\
&\le N'_{R+y,k-1}+N'_{R,k-1}+2^{k-1}\binom{d}{k-1}(S_{R+y,0}-S_{R,0}). \qquad (14)
\end{aligned}$$

Using the convergence $(C)_{k-1}$ and Lemma 2 applied to $k=0$, we deduce that

$$\frac{\#\{(k-1)\text{-faces intersecting } B(R+y)\setminus B(R)\}}{v(R)}\to 0 \text{ a.s. when } R\to+\infty. \qquad (15)$$

Combining inequalities (11) and (13), we get

$$\begin{aligned}
\frac{N'_{R,k}}{v(R)} \le\ & \frac{1}{A_k}\frac{S_{R+y,k}-S_{R,k}}{v(R)}\\
&+\frac{2(d-k+1)}{v(R)}\#\{(k-1)\text{-faces intersecting } B(R+y)\setminus B(R)\},
\end{aligned}$$

which by Lemma 2 and (15), implies the a.s. convergence $(C)_k$. This completes the proof of Lemma 3.

□

Remark 1 Lemma 1 implies that the rest $\varepsilon(R,\mathbf{1},\cdot)$ in (6) tends to zero a.s., which means

$$\frac{N_R}{v(R)}\to\mathbf{E}\left(\frac{1}{V_d(C_0)}\right) \quad \textit{a.s.} \quad \textit{when } R\to+\infty. \qquad (16)$$

Remark 2 In [1], we provided a way to define a random variable $\mathcal{C}$ (called the *typical cell*) with values in $\mathcal{K}$ such that for any translation-invariant, positive and bounded measurable function h on $\mathcal{K}$, we have

$$\mathbf{E}(h(\mathcal{C}))=\lim_{R\to+\infty}\frac{1}{N_R}\sum_{C\in\mathcal{C}_R}h(C),\ \textit{a.s.}. \qquad (17)$$

2 The thickened Poissonian tessellation.

We now introduce the notion of thickened Poissonian tessellation and we prove that the empirical distributions of this new tessellation are the same as in section 1.

Consider $e\ge 0$ and for all $x\in\mathbb{R}^d$, the polar hyperplane with thickness e at x by

$$H^e(x)=\{y\in\mathbb{R}^d; -e/2\le(y-x)\cdot x\le e/2\}.$$

We call the set

$$\mathcal{H}^e(\Phi) = \{H^e(x_i); x_i \in \Phi\}$$

a *thickened Poisson hyperplane process* in $\mathbb{R}^d$, and the closure of a connex component of the set

$$\mathcal{E}^e = \mathbb{R}^d \setminus \cup_{x_i \in \Phi} H^e(x_i)$$

a *cell* associated to $\mathcal{H}_e$.
Moreover, we define if $x \in \mathbb{R}^d$, C_x^e as the cell containing x if $x \notin \cup_{x_i \in \Phi} H^e(x_i)$ and $\emptyset$ else. As in the classical case, we can introduce the set $\mathcal{C}_R^e$ (resp. $\mathcal{C}_R^e{}'$) of cells included in $B(R)$ (resp. intersecting the boundary of $B(R)$) and $N_R^e = \#\mathcal{C}_R^e$ (resp. $N_R^e{}' = \#\mathcal{C}_R^e{}'$).

It is immediate that a.s., $N_R^e + N_R^e{}' \leq N_R + N_R'$, so N_R^e and $N_R^e{}'$ are integrable. Moreover, using Proposition 1, we have

$$\frac{N_R^e{}'}{v(R)} \leq \frac{N_R'}{v(R)} \to 0 \text{ a.s. when } R \to +\infty. \tag{18}$$

We prove in the following lemma that the empirical means on the thickened tessellation exist as in (9) and do not depend on the thickness $e \geq 0$:

Theorem 1 *Let $h : \mathcal{K} \longrightarrow \mathbb{R}_+$ be a translation-invariant, positive and bounded measurable function. Then:*
(i) $\lim_{R\longrightarrow\infty} \frac{1}{N_R^e} \sum_{C \in \mathcal{C}_R^e} h(C)$ *exists a.s. and is a constant.*
(ii) *This limit is independent of $e \geq 0$.*

Proof. **(i)** Following the case without thickness, we apply the equalities $C_0^e(T^{-a}\omega) = C_a^e(\omega) - a$, $a \in \mathcal{E}^e$, $\omega \in \Omega$, to get the identities:

$$\frac{1}{v(R)} \int_{B(R)\cap\mathcal{E}^e} \frac{dx}{V_d(C_0^e(T^{-x}\omega))} = \frac{N_R^e}{v(R)} + \frac{1}{v(R)} \varepsilon^e(R, \mathbf{1}, \omega) \tag{19}$$

$$\frac{1}{v(R)} \int_{B(R)\cap\mathcal{E}^e} \frac{h(C_0^e(T^{-x}\omega))}{V_d(C_0^e(T^{-x}\omega))} dx = \frac{1}{v(R)} \sum_{C \in \mathcal{C}_R^e} h(C) + \frac{1}{v(R)} \varepsilon^e(R, h, \omega), \tag{20}$$

where

$$\varepsilon^e(R, h, \cdot) = \sum_{C \in \mathcal{C}_R^e{}'} h(C) \frac{V_d(C \cap B(R))}{V_d(C)} \text{ a.s..}$$

Taking the expectation in (20) and using the integrability of $(N_R^e + N_R^e{}')$, we obtain as in the no-thickness case that

$$\mathbf{E}\left\{ \frac{\mathbf{1}_{\mathcal{E}^e}(0) h(C_0^e)}{V_d(C_0^e)} \right\} < +\infty. \tag{21}$$

Let us notice that the term $\varepsilon^e(R, h, \cdot)/v(R)$ tends to zero a.s. when R goes to infinity, by using (18) and the inequality $\varepsilon^e(R, h, \cdot) \leq K_h N_R^e{}'$.

Consequently, we can use Wiener's ergodic theorem, as in the preceding section to obtain that

$$\lim_{R\to+\infty} \frac{N_R^e}{v(R)} = \mathbf{E}\left\{ \frac{\mathbf{1}_{\mathcal{E}^e}(0)}{V_d(C_0^e)} \right\}, \quad \text{a.s.} \tag{22}$$

and

$$\lim_{R\longrightarrow\infty}\frac{1}{N_R^e}\sum_{C\in\mathcal{C}_R^e}h(C)=\left[\mathbf{E}\left\{\frac{\mathbf{1}_{\mathcal{E}^e}(0)}{V_d(C_0^e)}\right\}\right]^{-1}\mathbf{E}\left\{\frac{h(C_0^e)\mathbf{1}_{\mathcal{E}^e}(0)}{V_d(C_0^e)}\right\}\quad\text{a.s.}\tag{23}$$

(ii) Let us notice that it suffices to show that for all bounded and measurable function h,

$$\frac{1}{\mathbf{P}\{0\in\mathcal{E}^e\}}\mathbf{E}\left\{\frac{h(C_0^e)\mathbf{1}_{\mathcal{E}^e}(0)}{V_d(C_0^e)}\right\}=\mathbf{E}\left\{\frac{h(C_0^0)}{V_d(C_0^0)}\right\}.\tag{24}$$

Actually, we will deduce the equality of the two limits in (9) and (23) by applying (24) first to $h=1$, then to any h.

Noticing that

$$\{0\in\mathcal{E}^e\}=\{B(e/2)\cap\Phi=\emptyset\},\tag{25}$$

we deduce easily that Φ conditioned by the event $\{0\in\mathcal{E}^e\}$, is distributed as $\Phi_{e/2}$, where $\Phi_{e/2}$ is a Poisson point process of intensity measure $\mathbf{1}_{B(e/2)^c}d\mu$. Consequently,

$$\frac{\mathbf{E}\left\{\frac{h(C_0^e)}{V_d(C_0^e)}\mathbf{1}_{0\in\mathcal{E}^e}\right\}}{\mathbf{P}\{0\in\mathcal{E}^e\}}=\mathbf{E}\left\{\frac{h(C_0^e(\Phi_{e/2}))}{V_d(C_0^e(\Phi_{e/2}))}\right\}.\tag{26}$$

Besides, let us consider the transformation

$$\varphi_{e/2}:\left\{\begin{array}{lll}\mathbb{R}^d\setminus B(e/2) & \longrightarrow & \mathbb{R}^d\\ x & \longmapsto & x-\frac{e}{2}\frac{x}{||x||},\end{array}\right.$$

and remark that

$$C_0^e(\Phi_{e/2})=C_0(\overline{\Phi}),\tag{27}$$

where $\overline{\Phi}=\{\varphi_{e/2}(y_i);y_i\in\Phi_{e/2}\}$.
It is easy to verify that $\overline{\Phi}$ is distributed as Φ. Combining (26) with (27), it then shows the equality (24), and so completes the proof of the point (ii) of Theorem 1.

□

3 The empirical distribution of the inball radius.

In this section, we make an easy use of Theorem 1 to obtain the empirical law of the inball radius of the Poissonian tessellation in $\mathbb{R}^d$, following an idea of Miles [10] for the dimension two. Let us define $R_I(\mathcal{C})$ the inball radius of the typical cell of the Poissonian tessellation.

Theorem 2 *$R_I(\mathcal{C})$ is exponentially distributed, of parameter σ_d, where σ_d denotes the area of $\mathbb{S}^{d-1}$.*

Proof. We notice that a cell has its inball radius more than t, $t\geq 0$, if and only if it is not recovered when we thicken each hyperplane by $2t$. So, applying (17) to $h=\mathbf{1}_{\{R_I\geq t\}}$

and using the fact the N'_R is negligible compared to N_R a.s. when $R \to +\infty$, we obtain a.s.

$$\mathbf{P}\{R_I(\mathcal{C}) \geq t\} = \lim_{R\to+\infty} \frac{N_R^{2t}}{N_R}.$$

Moreover, combining (16), (22) and (24) applied to $h = 1$, we obtain

$$\begin{aligned}
\mathbf{P}\{R_I(\mathcal{C}) \geq t\} &= \lim_{R\to+\infty} \frac{{N_R}^{2t}}{N_R} \\
&= \left[\mathbf{E}\left(\frac{1}{V_d(C_0)}\right)\right]^{-1} \mathbf{E}\left(\frac{\mathbf{1}_{\mathcal{E}^{2t}}(0)}{V_d(C_0^{2t})}\right) \\
&= \mathbf{P}\{0 \in \mathcal{E}^{2t}\}. \qquad (28)
\end{aligned}$$

Using (25) and the Poissonian property of Φ, we have

$$\begin{aligned}
\mathbf{P}\{0 \in \mathcal{E}^{2t}\} &= \mathbf{P}\{\Phi \cap B(t) = \emptyset\} \\
&= e^{-\mu(B(t))} = e^{-\sigma_d t}. \qquad (29)
\end{aligned}$$

Inserting the equality (29) in (28), we deduce Theorem 2.

□

References

[1] P. Calka. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6):557–562, 2001.

[2] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *To appear in Adv. in Appl. Probab.*, 2002.

[3] R. Cowan. The use of the ergodic theorems in random geometry. *Adv. Appl. Probab.*, (suppl.):47–57, 1978. Spatial patterns and processes (Proc. Conf., Canberra, 1977).

[4] R. Cowan. Properties of ergodic random mosaic processes. *Math. Nachr.*, 97:89–102, 1980.

[5] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[6] A. Goldman. Sur une conjecture de D. G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne. *Ann. Probab.*, 26(4):1727–1750, 1998.

[7] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[8] S. Goudsmit. Random distribution of lines in a plane. *Rev. Modern Phys.*, 17:321–322, 1945.

[9] I. N. Kovalenko. A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons. *J. Appl. Math. Stochastic Anal.*, 12(4):301–310, 1999.

[10] R. E. Miles. Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci. U.S.A.*, 52:901–907, 1964.

[11] R. E. Miles. Random polygons determined by random lines in a plane. II. *Proc. Nat. Acad. Sci. U.S.A.*, 52:1157–1160, 1964.

[12] R. E. Miles. The various aggregates of random polygons determined by random lines in a plane. *Advances in Math.*, 10:256–290, 1973.

[13] K. Paroux. *Théorèmes centraux limites pour les processus poissoniens de droites dans le plan et questions de convergence pour le modèle booléen de l'espace euclidien.* PhD thesis, Univ. Lyon 1, 1997.

[14] K. Paroux. Quelques théorèmes centraux limites pour les processus Poissoniens de droites dans le plan. *Adv. in Appl. Probab.*, 30(3):640–656, 1998.

[15] N. Wiener. The ergodic theorem. *Duke Math.*, 5:1–18, 1939.

Chapitre 6

Modélisation stochastique d'un phénomène unidirectionnel de fissuration.

6.1 Introduction et présentation des résultats.

Les résultats présentés dans cette partie ont été obtenus en commun avec A. Mézin et P. Vallois.

La détermination de la loi du rayon R_M du plus petit disque centré à l'origine contenant la cellule typique de Poisson-Voronoi ou la cellule de Crofton dans le plan nous a conduit à étudier des problèmes de recouvrement du cercle par des arcs aléatoires indépendants identiquement distribués, dont les centres sont uniformément répartis sur le cercle et les longueurs suivent une loi ν fixée.

Il existe d'autres modes de recouvrement aléatoire (du cercle ou de la droite réelle) plus complexes. En particulier, le modèle connu dans la littérature sous le nom de *modèle des parkings* [46], [68], [73], se présente de la manière suivante : on dispose d'une suite infinie $\{\mathcal{A}_i\}_{i\geq 1}$ d'arcs de cercle de longueur fixée $2a > 0$ et dont les centres $\{X_i; i \geq 1\}$ sont i.i.d. de loi uniforme sur le cercle. On place $\mathcal{A}_1$, puis on décide que l'on place $\mathcal{A}_2$ si et seulement si son centre $X_2 \notin \mathcal{A}_1$. Dans le cas contraire, on rejette cet arc et l'on considère $\mathcal{A}_3$. L'algorithme de recouvrement à l'étape $n \geq 2$ consiste à placer l'arc $\mathcal{A}_n$ sur le cercle si $X_n \notin \cup_{i\in E_n}\mathcal{A}_i$, E_n désignant l'ensemble des indices d'arcs qui ont été effectivement placés sur le cercle entre les étapes 1 et $(n-1)$. Si la condition précédente n'est pas vérifiée, on élimine l'arc $\mathcal{A}_n$ et on pose $E_{n+1} = E_n$.

La complexité de modèle vient du fait que l'on impose un ordre d'apparition des arcs, contrairement au recouvrement i.i.d. de base présenté dans la partie 2. Autrement dit, il faut non seulement traiter l'aléatoire qui intervient dans la position des arcs mais également prendre en compte une donnnée temporelle. En général, le problème a été considéré sur un intervalle de longueur fixée $L > 0$. Les deux questions centrales soulevées par la définition du modèle des parkings sont :

(i) d'étudier le nombre $N(L)$ d'intervalles (qui subsistent) sur $[0, L]$ ainsi que leur répartition lorsqu'on poursuit l'algorithme à l'infini (c'est-à-dire jusqu'à saturation). Dans cette situation, tous les arcs retenus recouvrent complètement l'intervalle

$[0, L]$;

(ii) d'étudier le nombre d'arcs conservés sur $[0, L]$ lorqu'on interrompt l'algorithme à l'étape n, ainsi que leur répartition et la mesure de Lebesgue de la partie non occupée de l'intervalle $[0, L]$.

La plupart des travaux dans la littérature traite le problème (i). Notons qu'à saturation, les distances entre les centres des intervalles conservés sur $[0, L]$ sont comprises entre a et $2a$. Avec le choix $a = 1$, A. Renyi [75] a fourni en 1958 un développement asymptotique de $\mathbf{E}N(L)$ lorsque L tend vers l'infini.

Pour tout $n \geq 1$,

$$\mathbf{E}N(L) = cL - (1 - c) + O(1/L^n), \tag{6.1}$$

où

$$c = \int_0^{+\infty} \exp\left(-2\int_0^t \frac{1 - e^{-u}}{u} du\right) dt \approx 0.748. \tag{6.2}$$

La convergence (6.1) a été obtenue indépendamment par A. Dvoretzky et H. Robbins [20] qui ont de plus prouvé un théorème central-limite. P. E. Ney [68] a prolongé cette étude en considérant des longueurs d'intervalle aléatoires. Par ailleurs, D. Mannion [46] a prouvé un théorème limite (en probabilité) portant sur les lois empiriques des distances entre centres d'intervalles successifs. Plus récemment, M. D. Penrose [73] a obtenu une extension de ces résultats dans un cadre multidimensionnel.

En revanche, le problème (ii) a été peu abordé jusqu'à présent. La question de la loi du nombre d'intervalles conservés sur $[0, L]$ (ou le cercle) lorsqu'on pousse l'algorithme jusqu'à l'étape $n \geq 1$ est encore ouverte. On ne sait d'ailleurs même pas déterminer l'espérance de cette variable (voir cependant en annexe 7.3 une piste possible de calcul). Le travail essentiel sur la question (ii) remonte à 1966 et est dû à B. Widom [92]. Celui-ci a montré par des méthodes plus ou moins heuristiques que le nombre moyen des distances de longueur donnée $(l + dl)$ entre centres d'intervalles successifs sur $[0, L]$ satisfait une équation différentielle qu'il a résolue, lorsque n, L tendent vers l'infini, et n/L étant fixé.

Dans le travail qui suit, nous proposons une généralisation du modèle des parkings à $\mathbb{R}$ tout entier qui permet de retrouver le résultat de Widom et d'obtenir de nouveaux calculs explicites sur les lois conjointes des distances entre deux centres d'intervalles successifs. Plus précisément, nous définissons un processus stationnaire Λ_ε sur la droite réelle ayant la propriété suivante : l'intersection de Λ_ε avec l'intervalle $[0, L]$ est identique en loi à l'ensemble des centres d'un modèle des parkings lorsqu'on interrompt l'algorithme décrit plus haut à une étape aléatoire, de loi de Poisson de moyenne $LF(\varepsilon)$, F étant une fonction dérivable croissante fixée.

L'intérêt de disposer d'un processus stationnaire sur la droite toute entière est que l'on peut utiliser tous les outils qui existent déjà pour l'étudier statistiquement : l'intensité (nombre moyen de points par unité de longueur) et la notion de distance typique entre deux points successifs au sens de Palm.

Pour aboutir à l'étude du processus Λ_ε, notre approche de départ a été de modéliser un phénomène de fissuration. Plus précisément, considérons un ensemble composé d'un substrat et d'un dépôt d'épaisseur négligeable (par exemple, une couche de peinture sur un mur ou de vase sur un étang). Lorsqu'on applique une force de traction unilatérale d'intensité ε à l'ensemble, le dépôt se craquèle et les fissures formées sont toutes parallèles.

Aussi, il suffit de considérer leurs projections sur un axe fixé, ce qui nous ramène à l'étude d'un processus ponctuel unidimensionnel.

Prenant pour hypothèse que la formation d'une fissure se fait indépendamment des fissures déjà existantes, A. Mézin et P. Vallois [53] ont fourni comme premier modèle pour l'ensemble des couples (position d'une fissure, niveau de contrainte exact auquel celle-ci se crée) un processus de Poisson Φ sur $\mathbb{R} \times \mathbb{R}_+$ de mesure d'intensité $f(y)\mathbf{1}_{\mathbb{R}_+}(y)dxdy$, où $f = F'$ est une fonction dépendant des données physiques du système étudié.

En fait, un certain nombre de travaux physiques [5], [25], [45], [52] montre que les fissures ne se forment pas indépendamment les unes des autres car la création d'une fissure sur le dépôt implique la relaxation de la contrainte sur tout un voisinage de la position. Par conséquent, on se donne pour hypothèse dans l'ensemble du travail le fait suivant : lorsqu'il existe une fissure en un point $x \in \mathbb{R}$, aucune nouvelle fissure ne peut se former par la suite dans l'intervalle $[x - r, x + r]$, où $r > 0$ est une constante fixée.

Suivant cette hypothèse, on construit le processus bi-dimensionnel Ψ sur $\mathbb{R} \times \mathbb{R}_+$ des couples (position d'une fissure, niveau de contrainte auquel celle-ci apparaît) en effaçant par une procédure bien précise certains points du processus Φ. On obtient que l'ensemble des positions des fissures, lorsqu'on applique une force d'intensité ε, est modélisé par le processus stationnaire et ergodique

$$\Lambda_\varepsilon = \{x \in \mathbb{R}; \exists y \in [0, \varepsilon] \mid (x, y) \in \Psi\}.$$

Tout d'abord, nous faisons une étude statistique du processus Λ_ε. On considère λ_ε l'intensité de Λ_ε et le couple $(D_\varepsilon, L_\varepsilon)$ constitué de la distance inter-fissures typique et du niveau de contrainte typique associé définis au sens de Palm. Nous déterminons l'expression explicite de λ_ε et de la loi conjointe de $(D_\varepsilon, L_\varepsilon)$ en résolvant une équation intégrale. Nous retrouvons en particulier les résultats que Widom avait obtenu dans un contexte moins précis mathématiquement.

De plus, nous construisons le processus $\Lambda_\varepsilon^+ = \{X_n; n \geq 1\}$ sur $\mathbb{R}_+$ (modélisant l'ensemble des fissures sur un dépôt unilatéral) par le même procédé que Λ_ε. Nous déterminons la loi de Λ_ε^+ point par point : en particulier, nous calculons explicitement la loi conjointe des positions des n premières fissures $X_1, \cdots, X_n$ et de leurs niveaux de contrainte associés, $Y_1, \cdots, Y_n$, $n \geq 1$. Nous en déduisons par des arguments utilisant les résultats de convergence à l'équilibre des chaînes de Harris que (X_n, Y_n) converge en loi vers $(D_\varepsilon, L_\varepsilon)$. De plus, le vecteur $(X_1, \cdots, X_n)$ suit la loi des n premiers points d'un processus de renouvellement explicite conditionné par un évènement que nous explicitons également.

Enfin, en faisant tendre ε vers l'infini, nous retrouvons l'ensemble des résultats énoncés ci-dessus. En particulier, λ_ε converge vers la constante de Renyi c fournie par l'égalité (6.2). Ainsi, notre modèle permet d'unifier dans une certaine mesure les approches des problèmes distincts (i) et (ii) présentés au début de cette introduction.

Stochastic modelling of a unidirectional multicracking phenomenon.*

Pierre Calka[†], André Mézin[‡] and Pierre Vallois[§]

Abstract

We work out a stationary process on the real line to model the unidirectional multiple cracking phenomenon that is observed in certain composites materials (coating on a ductile substrate, brittle fibres in a matrix,etc) when submitted to a uniaxial strain. The position X_i^ε of the crack and the value Y_i^ε of the strain ε at which it forms constitute the two coordinates of the bi-dimensional point process under consideration. The stress relaxation around the cracks is taken into account thanks to an adequate erasing procedure, so that no new crack is allowed to form within a distance r around every existing crack. For a fixed strain ε, we calculate the intensity of the process and the distribution of the intercrack distance in the Palm sense. An explicit expression of the quantities of interest can be obtained. Another point of view is developed, where the points X_i^ε are described from a fixed origin. The distribution of $\{(X_i^\varepsilon, Y_i^\varepsilon), 1 \leq i \leq n\}$ is a conditioned renewal process. The approaches "in the Palm sense" and "fixed origin" merge for $n \to +\infty$.

Introduction.

Situations where the presence of many different cracks can be observed in the same structure are not rare. Geology provides numerous examples of three-dimensional crack network [4],[7]. Mud cracking, crazing of ceramics or cracks in paintings [3] are typical familiar examples of two-dimensional networks. Cracking of coatings, which is of particular interest in material science, also may be considered a two-dimensional phenomenon [9].

In this work, we consider the particular case in which the crack network consists of rather long and parallel cracks [2],[6],[11], which may be analyzed as a one-dimensional

American Mathematical Society 2000 subject classifications. Primary 60G10, 60G55, 60K40; secondary 60B10, 60J05, 60K15.

Key words and phrases. brittleness, coating, composite material, crack, fibre, Markotheseart7v chain, Palm measure, Poisson point process, relaxation of stress, renewal process, rupture, stationary processes.

[†] *Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier, 69366 Lyon Cedex 07, France. *E-mail*: Pierre.Calka@univ-lyon1.fr

[‡] *Postal address*: Laboratoire de Sciences et Génie des Surfaces, UMR CNRS 7570, Ecole des Mines, Parc de Saurupt, 54042 Nancy Cedex, France. *E-mail*: Andre.Mezin@mines.u-nancy.fr

[§] *Postal address*: Institut Élie Cartan, Université Henri Poincaré, Nancy I B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex, France. *E-mail*: Pierre.Vallois@iecn.u-nancy.fr

problem. This situation may result from a natural cause [2] or correspond with a particular mechanical sollicitation [6],[11] of a given physical system. Consider for instance the multiple cracking test of brittle coatings. A uniaxial, regularly increasing strain (elongation), denoted ε, is applied to a specimen consisting of a substrate covered with the coating to be studied, which results in the formation of parallel cracks in the coating, orthogonal to the strain direction [6],[11]. Consequently, the geometrical aspect of the problem practically reduces to the intersections of the cracks with a predefined line in the direction of the strain axis.

Mézin and Vallois [12] considered the purely theoretical case where an existing crack does not influence the formation of the new ones. Fixing the specimen length to $L > 0$, they described the multiple cracking phenomenon through a Poisson point process on $[0, L]$, of intensity $F(\varepsilon)$, where F is a positive, increasing and derivable function depending on the physical properties of the coating [1]. They also represented the set of the couples formed by the position of a crack and the exact strain level at which it appeared by a Poisson point process on $[0, L] \times \mathbb{R}_+$, of intensity measure $\mathbf{1}_{\mathbb{R}_+}(y)f(y)dxdy$, where $f = F'$.

In actual fact, the cracks do not form independently from each other, because the formation of a crack in the coating results in a relaxation of the stress in the vicinity of the crack [2],[6],[9],[11]. So no new crack can form close to an existing crack because the stress is very low in this area. We here propose a stochastic model that takes into account this phenomenon of relaxation of stress.

Denoting by $L > 0$ the length of the coating, we assume that no new crack can form in an interval of length $r > 0$ around each existing crack. We then take on the one hand a sequence $\{X_i; i \geq 1\}$ of independent and uniformly distributed variables on the segment $[0, L]$ and on the other hand a Poisson variable denoted N, of mean value $LF(\varepsilon)$, independent of the preceding sequence. We then throw successively $X_1, \cdots, X_N$ on the segment but keeping only some of them according to the following procedure. Suppose $N \geq 2$, we take X_1 and after we erase X_2 if and only if X_2 is in the interval of radius r around X_1. Having decided if $X_2, \cdots, X_n$ are kept or not, for a fixed $n \geq 2$, such that $n \leq N - 1$, we erase X_{n+1} if it belongs to the union of all the intervals.

We observe that this well-defined model is difficult to deal with, particularly in the absence of any idea about the law of the number of the preserved cracks.

Instead of throwing a Poissonian number of uniform variables, Rényi [17] in 1958 worked out a model where cracks are placed on the segment up to saturation. He obtained the asymptotic behaviour of the mean number of cracks on $[0, L]$, when L goes to infinity. This question, known as the car-parking problem, has been largely investigated (see for example [10], [15], [16]).

Fixing the number N of thrown variables, Widom [19] in 1966 demonstrated by heuristic methods that the mean number of inter-crack distance of given length satisfies a differential equation and provided formulas for the empirical distribution function of the inter-crack distances when $N, L \to +\infty$, with N/L fixed.

The stochastic approach that we propose in this paper is based on previous results [12] related to the simple situation of no stress relaxation. As in [12], we start with a Poisson process Φ on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $\mathbf{1}_{\mathbb{R}_+}(y)f(y)dxdy$. An erasing procedure is defined so that every crack closer that r from an existing crack is eliminated, which leads to a countable subset Ψ of Φ. The two-dimensional character of the problem, rather than being a complication, here is an asset towards the solution. A point (X, Y) of Ψ corresponds to the position X of the crack and to the strain Y at which it forms respectively. For every given $\varepsilon > 0$,

$$\Lambda_\varepsilon = \{x \in \mathbb{R}; \exists y \in [0, \varepsilon] \mid (x, y) \in \Psi\},$$

is the projection on the x-axis of $\Psi \cap (\mathbb{R} \times [0, \varepsilon])$.

In section 2, we demonstrate that the process Λ_ε is stationary. In particular, the mean crack number λ_ε, the couple $(D_\varepsilon, L_\varepsilon)$ of the inter-crack distance and the strain level in the Palm sense are defined precisely, and another different notion of inter-crack distance I_0^ε is given. The results are expressed through two unknown functions G and H.

We demonstrate in section 3 that the function G satisfies an integral equation that can be solved, which allows in turn the function H to be determined. An explicit expression of λ_ε and of the law of $(D_\varepsilon, L_\varepsilon)$ (resp. I_0^ε) thus can be obtained.

For getting a better understanding of the process Λ_ε, an alternative point of view is considered in section 4, that consists in describing the points X_i^ε from a fixed origin

$$0 < X_1^\varepsilon < X_2^\varepsilon < \cdots < X_n^\varepsilon, \quad n \geq 1.$$

It appears that, due to the complexity of the erasing procedure, $(X_n^\varepsilon)_{n\geq 1}$ is not a renewal process, but a conditioned renewal process. The two approaches "in the Palm sense" and the present "fixed origin" merge in that $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$ converges in law as $n \to +\infty$, to a r.v. of same law as $(D_\varepsilon, L_\varepsilon)$.

Section 5 presents the saturation case already considered by Renyi. Having defined the process Λ_∞ associated to saturation, we demonstrate that λ_ε tends to λ_∞ as $\varepsilon \to +\infty$. In the same way, D_ε (resp. I_0^ε, L_ε) converges in distribution to D_∞ (resp. I_0^∞, L_∞). We also give a complete description of the law of $(X_n^\infty, Y_n^\infty)_{n\geq 1}$ and a result of convergence in law of $(X_{n+1}^\infty, Y_n^\infty)$ to (D_∞, L_∞).

1 A stationary model with relaxation of stress.

In this section, we define a stationary process Λ_ε on $\mathbb{R}$ that for a given strain ε represents the positions of the cracks, in the case where the stress is relaxed on an interval of radius $r > 0$ around every existing crack.

To this end, we introduce a two-dimensional point process Ψ on $\mathbb{R} \times \mathbb{R}_+$ such that the first and the second coordinates of a point Ψ represent the position of a crack and the strain level at which the crack formed respectively. Λ_ε is the projection on the x-axis of $\Psi \cap (\mathbb{R} \times [0, \varepsilon])$:

$$\Lambda_\varepsilon = \{x \in \mathbb{R}; \exists y \in [0, \varepsilon] \mid (x, y) \in \Psi\}. \tag{1}$$

Considering a two-dimensional point process is a convenient way to order the cracking positions as in the case of the segment $[0, L]$, by associating with any position an "arrival time" of the crack. To define Ψ, we start with the process Φ associated with the cracking phenomenon without stress relaxation. Φ is a Poisson point process on $\mathbb{R}\times\mathbb{R}_+$, of intensity measure ν:

$$\nu(dx, dy) = \mathbf{1}_{\mathbb{R}_+}(y) f(y) dx dy, \tag{2}$$

where $f = F'$ [12].

Since ν is absolutely continuous with respect to the Lebesgue measure, then a.s. if (x, y) and (x', y') are two points in Φ, y is not equal to y'.

For any point $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, we define the corresponding domain of relaxation:

$$R(x, y) = [x - r, x + r] \times [y, +\infty) \subset \mathbb{R} \times \mathbb{R}_+.$$

We construct Ψ as a sub-process of Φ by using the following recursive algorithm.
Initialization. We start with taking any couple (x, y) in Ψ, such that y is a local minimum, i.e.

$$\Phi \cap ([x - r, x + r] \times [0, y)) = \emptyset.$$

Let us denote by Ψ_1 the set of these points and by Φ_1 the subset of Φ obtained by erasing all the points that are in the domains of relaxation associated to the points of Ψ_1. This means

$$\Phi_1 = \Phi \cap \left(\cup_{(x,y)\in\Psi_1} R(x, y)\right)^c.$$

Iteration. Suppose that for a fixed $n \in \mathbb{N}^*$, the processes $\Phi_1, \cdots, \Phi_n$ and $\Psi_1, \cdots, \Psi_n$ are constructed.
We then take in Ψ_{n+1} the points (x, y) of Φ_n such that y is a local minimum. We define Φ_{n+1} as the set of the points of Φ_n not erased by the domains of relaxation associated to the points of Ψ_{n+1}. In mathematical terms,

$$\begin{cases} \Psi_{n+1} = \{(x, y) \in \Phi_n ; \Phi_n \cap [x - r, x + r] \times [0, y) = \emptyset\} \\ \Phi_{n+1} = \Phi_n \cap \left(\cup_{(x,y)\in\Psi_{n+1}} R(x, y)\right)^c \end{cases}$$

We then define

$$\Psi = \cup_{n\geq 1} \Psi_n. \tag{3}$$

From now on the points of Ψ will be named *erasers*, and the points of $\Phi \setminus \Psi$ that are deleted by the domains of relaxation associated to the erasers, will be named *erased*. So,

$$\Phi \setminus \Psi = \{(x, y) \in \Phi ; \exists (x', y') \in \Psi \mid (x, y) \in R(x', y')\}.$$

The point process Ψ can also be seen as the complementary set in Φ of the *erasing tree* $\mathcal{A}(\Phi)$, where

$$\mathcal{A}(\Phi) = \cup_{(x,y)\in\Psi}([x - r, x + r] \times (y, +\infty)). \tag{4}$$

We say that any point of $\mathbb{R} \times \mathbb{R}_+$ is *erased* if it is contained in the erasing tree $\mathcal{A}(\Phi)$.

The first properties of Ψ are stated in the following proposition.

Proposition 1 **(i)** *Almost surely the projections of the points of Ψ on the x-axis are separated by a distance at least equal to r.*
(ii) Ψ *is infinite a.s..*
(iii) Ψ *is invariant under horizontal translations.*
(iv) Ψ *is ergodic.*

Proof. (i) Let us consider two points $(x, y), (x', y') \in \Psi$ and suppose that

$$(x, y) \in \Psi_n \text{ and } (x', y') \in \Psi_m, \quad m \geq n.$$

Then $(x', y') \notin R(x, y)$, so $|x' - x| > r$.
(ii) It suffices to show that Ψ_0 is infinite. Let us denote C_n, for every $n \in \mathbb{Z}$, the event "the minimum of the second coordinates of the points of $\Phi \cap [3nr, 3(n+1)r] \times \mathbb{R}_+$ is reached at a point of $[(3n+1)r, (3n+2)r] \times \mathbb{R}_+$".

Let us remark that

$$C_n \subset \{\Psi_0 \cap [(3n+1)r, (3n+2)r] \times \mathbb{R}_+ \neq \emptyset\}, \quad n \in \mathbb{Z}.$$

Since Φ is a Poisson point process, the events C_n are mutually independent and have the same positive probability. So using Borel-Cantelli's lemma,

$$\mathbf{P}\{\limsup C_n\} = 1,$$

which proves that Ψ_0 is infinite.
(iii) Let us consider the set $\mathcal{M}_\sigma(\mathbb{R}^2)$ of the locally finite sequences of $\mathbb{R}^2$, endowed with the $\sigma-$field generated by the applications $\phi \longmapsto \#(\phi \cap A)$, $\phi \in \mathcal{M}_\sigma(\mathbb{R}^2)$, where $A \in \mathcal{B}(\mathbb{R}^2)$. We define for any $x \in \mathbb{R}$,

$$T^x : \begin{cases} \mathcal{M}_\sigma(\mathbb{R}^2) & \longrightarrow \quad \mathcal{M}_\sigma(\mathbb{R}^2) \\ \{(x_i, y_i)\}_{i \geq 1} & \longmapsto \quad \{(x_i + x, y_i)\}_{i \geq 1} \end{cases}$$

Then it immediately appears that like for Φ, Ψ is invariant in law under the applications T^x, $x \in \mathbb{R}$.
(iv) To prove the ergodicity, it suffices to show that Ψ is strongly mixing for the applications T^x, which means that for any bounded Borel sets A, B of $\mathbb{R} \times \mathbb{R}_+$, and all $k, l \geq 0$, when $|x|$ goes to infinity,

$$\mathbf{P}[\{\#(\Psi \cap A) = k\} \cap \{\#(T^{-x}(\Psi) \cap B) = l\}] \longrightarrow \mathbf{P}\{\#(\Psi \cap A) = k\} \cdot \mathbf{P}\{\#(\Psi \cap B) = l\}. \quad (5)$$

Since Ψ is invariant by $T^{x/2}$, we have

$$\begin{aligned} &\mathbf{P}\{(\#(\Psi \cap A) = k) \cap (\#(T^{-x}(\Psi) \cap B) = l)\} \\ &\qquad = \mathbf{P}\{(\#(T^{x/2}(\Psi) \cap A) = k) \cap (\#(T^{-x/2}(\Psi) \cap B) = l)\} \\ &\qquad = \mathbf{P}\{(\#(\Psi \cap (A - x/2)) = k) \cap (\#(\Psi \cap (B + x/2)) = l)\}. \quad (6) \end{aligned}$$

In order to prove the asymptotic independence of $\{\#(\Psi \cap (A - x/2)) = k\}$ and $\{\#(\Psi \cap (B + x/2)) = l\}$, we are going to express these two events with the two independent processes Φ^+ and Φ^- where

$$\Phi^+ = \Phi \cap (\mathbb{R}_+ \times \mathbb{R}_+) \quad \text{and} \quad \Phi^- = \Phi \cap (\mathbb{R}_- \times \mathbb{R}_+). \quad (7)$$

Let Z_+ (respectively Z_-) be the minimum (respectively the maximum) of the first coordinates of the points of Ψ_0 contained in the domain $[r, +\infty) \times \mathbb{R}_+$ (respectively $(-\infty, -r] \times \mathbb{R}_+$). In other words, Z_- (respectively Z_-) is the minimal (respectively maximal) first coordinate of the points of Φ^+ (respectively Φ^-) such that $\Phi^+ \cap [x-r, x+r] \cap [0, y) = \emptyset$ (respectively $\Phi^- \cap [x-r, x+r] \cap [0, y) = \emptyset$). So Z_+ and Z_- are two independent variables.

Besides, let us remark that defining the erasing trees $\mathcal{A}(\Phi^+)$ and $\mathcal{A}(\Phi^-)$ as for Φ, we have

$$\mathcal{A}(\Phi^+) \cap ([Z_+, +\infty) \times \mathbb{R}_+) = \mathcal{A}(\Phi) \cap ([Z_+, +\infty) \times \mathbb{R}_+), \quad (8)$$

and

$$\mathcal{A}(\Phi^-) \cap ((-\infty, Z_-] \times \mathbb{R}_+) = \mathcal{A}(\Phi) \cap ((-\infty, Z_-] \times \mathbb{R}_+). \qquad (9)$$

For $n \in \mathbb{N}$, let x be such that $x/2 + \inf p_1(B) > n$ and $-x/2 + \sup p_1(A) < -n$, where $p_1(A)$ denotes the projection on the x-axis of A. Let us consider the events

$$E_n^+ = \{Z_+ \in [0, n]; \#(\Psi \cap (B + x/2)) = l\}$$

and

$$E_n^- = \{Z_- \in [-n, 0]; \#(\Psi \cap (A - x/2)) = k\}.$$

Then E_n^+ and E_n^- are independent because the equalities (8) and (9) imply that

$$E_n^+ = \{Z_+ \in [0, n]; \#(\Phi^+ \cap [\mathcal{A}(\Phi^+) \cap [Z_+, +\infty) \times \mathbb{R}_+]^c \cap (B + x/2)) = l\},$$

and

$$E_n^- = \{Z_- \in [-n, 0]; \#(\Phi^- \cap [\mathcal{A}(\Phi^-) \cap (-\infty, Z_-] \times \mathbb{R}_+]^c \cap (A - x/2)) = l\}.$$

Consequently, let us fix $\eta > 0$ and choose $n \in \mathbb{N}$ such that

$$\mathbf{P}\{Z_+, Z_- \in [0, n]\} \geq 1 - \eta/3.$$

Then for $x \geq 2 \sup\{(n - \inf p_1(B)), (n + \sup p_1(A))\}$, using the invariance of Ψ under $T^{x/2}$ and the independence of E_n^+ and E_n^-, we have

$$\begin{aligned}
&|\mathbf{P}\{(\#(\Psi \cap A) = k) \cap (\#(T^{-x}(\Psi) \cap B) = l)\} - \mathbf{P}\{\#(\Psi \cap A) = k\} \cdot \mathbf{P}\{\#(\Psi \cap B) = l\}| \\
&\quad \leq |\mathbf{P}\{(\#(\Psi \cap (A - x/2)) = k) \cap (\#(\Psi \cap (B + x/2)) = l)\} - \mathbf{P}\{E_n^+ \cap E_n^-\}| \\
&\qquad\qquad + |\mathbf{P}(E_n^+)\mathbf{P}(E_n^-) - \mathbf{P}\{\#(\Psi \cap (A - x/2)) = k\}\mathbf{P}\{\#(\Phi \cap (B + x/2)) = l\}| \\
&\quad \leq \frac{\eta}{3} + 2 \cdot \frac{\eta}{3} = \eta.
\end{aligned}$$

So the required convergence (5) is proved.

□

2 The mean crack number and typical inter-crack distance.

Let us consider for a fixed $\varepsilon > 0$, the set Λ_ε of the cracks given by the equality (1). Due to Proposition 1, Λ_ε is stationary and ergodic. As explained in the introduction, we can define directly the formation of cracks on a fixed interval $[0, L]$. Unfortunately, calculations in this setting are untractable.

We are interested in two physical quantities, the mean crack number and typical inter-crack distance that using the stationarity of Λ_ε we are able to define precisely. First, the mean crack number λ_ε, i.e. the mean number of cracks per unit of length is the intensity of Λ_ε. Secondly, we can define two different characteristic distances:
(i) the typical inter-crack distance D_ε is the "empirical mean" of points in Λ_ε, namely it

is the mean, in the Palm sense, of the distance between two successive cracks.
(ii) for any $x \in \mathbb{R}$, let us denote by I_x^ε the smallest interval whose bounds are in Λ_ε and that contains x (it is unique almost surely). Since Λ_ε is stationary, the distribution of $|I_x^\varepsilon|$ does not depend on x.

In this section we start with a precise definition of λ_ε and D_ε. We express the distribution of $|I_0^\varepsilon|$ via the law of D_ε. In particular, D_ε and $|I_0^\varepsilon|$ are not identically distributed. This leads us to determine λ_ε and the distribution of D_ε. In a first step we prove that λ_ε and the probability distribution function of D_ε write down as an integral of two functions G and H. The calculation of G and H is postponed in section 3.

Let us define λ_ε and D_ε. The application that associates to any Borel set $B \subset \mathbb{R}$, the value $\mathbf{E}(\#(\Lambda_\varepsilon \cap B))$ is a measure invariant by translations, so it is proportional to the Lebesgue measure on $\mathbb{R}$ denoted by $|\cdot|$. Consequently, we can define the intensity λ_ε of the process Λ_ε by the equality

$$\lambda_\varepsilon = \frac{1}{|B|}\mathbf{E}(\#(\Lambda_\varepsilon \cap B)), \tag{10}$$

where $B \subset \mathbb{R}$ is a fixed Borel set verifying $0 < |B| < +\infty$.
The law in the Palm sense of the typical inter-crack distance D_ε is defined as follows (see section 3 of [13] or section 2 of [14] for a complete survey on Palm distributions of stationary point processes on the real line).

For any measurable function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and all fixed Borel set $B \subset \mathbb{R}$,

$$\mathbf{E}h(D_\varepsilon) = \frac{1}{\lambda_\varepsilon |B|}\mathbf{E}\left\{\sum_{x\in\Lambda_\varepsilon\cap B} h(v(x,\Lambda_\varepsilon) - x)\right\}, \tag{11}$$

where

$$v(x,\Gamma) = \inf\{\Gamma \cap (x,+\infty)\} = \inf\{s \in \Gamma; s > x\}, \quad x \in \mathbb{R}, \tag{12}$$

for any non-empty subset Γ of $\mathbb{R}$, with the convention $\inf \emptyset = +\infty$.

We deduce immediately from Proposition 1 that

$$\mathbf{P}\{D_\varepsilon \geq r\} = 1.$$

Besides, using the same argument as Møller in [13], page 62, we obtain that D_ε is integrable and

$$\mathbf{E}D_\varepsilon = \frac{1}{\lambda_\varepsilon}. \tag{13}$$

We now establish a connection between the distributions of D_ε and $|I_0^\varepsilon|$. Recall that I_x^ε is the smallest interval containing x and whose bounds are elements of Λ_ε. Let $\mathcal{I}_L^\varepsilon$, $L > 0$, be the set of the intervals I_x^ε that are totally included in the segment $[-L, L]$, and N_L^ε the cardinal of $\mathcal{I}_L^\varepsilon$, i.e.

$$N_L^\varepsilon = \#\mathcal{I}_L^\varepsilon = \left(\#\{\Lambda_\varepsilon \cap [-L, L]\} - 1\right)_+.$$

The following proposition connects D_ε to the interval I_0^ε containing 0, and also to the empirical means of the intervals of $\mathcal{I}_L^\varepsilon$.

Proposition 2 **(i)** *For any positive measurable function h on $\mathbb{R}_+$,*

$$\mathbf{E}h(D_\varepsilon) = \frac{1}{\mathbf{E}(1/|I_0^\varepsilon|)}\mathbf{E}\left(\frac{h(|I_0^\varepsilon|)}{|I_0^\varepsilon|}\right). \tag{14}$$

(ii) *If there exists $p > 1$ such that $\mathbf{E}(h(D_\varepsilon)^p) < +\infty$, then when L goes to infinity,*

$$\frac{1}{N_L^\varepsilon}\sum_{I\in\mathcal{I}_L^\varepsilon} h(|I|) \longrightarrow \mathbf{E}h(D_\varepsilon) \quad \text{a.s.} \tag{15}$$

Proof. (i) It suffices to combine (13) with the argument used by Møller in [13], Prop. 3.3.2., for the typical cell of a Voronoi tessellation on $\mathbb{R}$ generated by a stationary point process.
(ii) Let us denote for all $x \in \mathbb{R}$,

$$\widetilde{T}^x : \left\{ \begin{array}{lcl} \mathcal{M}_\sigma(\mathbb{R}) & \longrightarrow & \mathcal{M}_\sigma(\mathbb{R}) \\ \{x_i\}_{i\geq 1} & \longmapsto & \{x_i + x\}_{i\geq 1}, \end{array} \right.$$

where $\mathcal{M}_\sigma(\mathbb{R})$ is the set of locally finite sequences of $\mathbb{R}$.

According to Wiener's ergodic theorem [20], if $\mathbf{E}(h(|I_0^\varepsilon|)/|I_0^\varepsilon|) < +\infty$, then when L goes to infinity,

$$\frac{1}{2L}\int_{-L}^{L}\frac{h(|I_x^\varepsilon|))}{|I_x^\varepsilon|}dx = \frac{1}{2L}\int_{-L}^{L}\frac{h(|I_0^\varepsilon(\widetilde{T}^{-x}(\Lambda_\varepsilon))|)}{|I_0^\varepsilon(\widetilde{T}^{-x}(\Lambda_\varepsilon))|}dx \longrightarrow \mathbf{E}\left(\frac{h(|I_0^\varepsilon|)}{|I_0^\varepsilon|}\right) \quad \text{a.s.} \tag{16}$$

Moreover, by taking $h = 1$, we easily verify that

$$\frac{N_L^\varepsilon}{2L} \longrightarrow \mathbf{E}\left(\frac{1}{|I_0^\varepsilon|}\right), \qquad \text{a.s. when } L \to +\infty.$$

We suppose that h satisfies the condition (ii). Applying the argument used by Goldman in the case of Poissonian tessellations (see [8], lemma 4), we demonstrate that

$$\frac{1}{2L}\int_{-L}^{L}\frac{h(|I_x^\varepsilon|)}{|I_x^\varepsilon|}dx - \frac{1}{2L}\sum_{I\in\mathcal{I}_L^\varepsilon} h(|I|) \longrightarrow 0, \qquad \text{when } L \to +\infty.$$

□

λ_ε and D_ε are two characteristics that describe the statistical properties of the process Λ_ε and are useful in applications. So from now on, our goal is to determine λ_ε and the law of D_ε.

For this purpose, the following lemma is an essential intermediate result.

Lemma 3 **(i)** $\lambda_\varepsilon = \int_0^\varepsilon \mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi)\}f(v)dv.$

(ii) *For every $t \geq 0$,*

$$\mathbf{P}\{D_\varepsilon \geq t\} = \frac{1}{\lambda_\varepsilon}\int_0^\varepsilon \mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi); v(0,\Lambda_{\varepsilon,v}) \geq t\}f(v)dv,$$

where $\Lambda_{\varepsilon,v}$ is the cracking process based on $\Phi \cap (\mathbb{R}\times[0,\varepsilon]) \cup \{(0,v)\}$ and $v(0,\Lambda_{\varepsilon,v})$ is defined by the equality (12).

Proof. In order to prove these two equalities, the essential tool is Slivnyak's formula (see for example [13]) satisfied by Φ:
For any positive measurable function h defined on $\mathbb{R}^2 \times \mathcal{M}_\sigma(\mathbb{R}^2)$, we have

$$\mathbf{E}\left\{\sum_{(x,y)\in\Phi} h((x,y),\Phi)\right\} = \int \mathbf{E}(h((u,v),\Phi\cup\{(u,v)\}))\nu(du,dv). \quad (17)$$

Let us fix $L > 0$. Then using successively (1), (10) and (17), we have

$$\begin{aligned}
\lambda_\varepsilon &= \frac{1}{L}\mathbf{E}\left[\#(\Lambda_\varepsilon \cap [0,L])\right] \\
&= \frac{1}{L}\mathbf{E}\left[\sum_{(x,y)\in\Phi} \mathbf{1}_{[0,L]}(x)\mathbf{1}_{[0,\varepsilon]}(y)\mathbf{1}_{\mathcal{A}(\Phi)^c}(x,y)\right] \\
&= \frac{1}{L}\int_0^L du \int_0^\varepsilon \mathbf{P}\{(u,v) \notin \mathcal{A}(\Phi\cup\{(u,v)\})\} f(v)dv \\
&= \int_0^\varepsilon \mathbf{P}\{(0,v)\notin \mathcal{A}(\Phi)\} f(v)dv,
\end{aligned}$$

the last equality resulting from the invariance under horizontal translations of Φ and from the equality between the two events $\{(0,v) \notin \mathcal{A}(\Phi\cup\{(0,v)\}\}$ and $\{(0,v)\notin\mathcal{A}(\Phi)\}$.

The result (i) then is demonstrated.

The same is done for (ii) by using the equality (11) defining D_ε in law and formula (17).

□

Let us consider the continuous function

$$G(x,y) = \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) \subset \mathcal{A}(\Phi^+)\}, \quad x\in[0,r], y\geq 0, \quad (18)$$

where Φ^+ is defined by (7).

$G(x,y)$ represents the probability that the points of the rectangle $[0,x]\times[0,y]$ contained in Φ are erased by erasers from the right (i.e. belonging to Φ^+).

More generally, we define the continuous function

$$\begin{aligned}
H(x,y,x',y') &= \mathbf{P}\{\Phi^+\cap([0,x]\times[0,y]\setminus[0,x']\times[y',y]))\subset\mathcal{A}(\Phi^+)\}, \\
&\qquad 0\leq x'\leq x\leq r, 0\leq y'\leq y. \quad (19)
\end{aligned}$$

$H(x,y,x',y')$ is the probability that the points of the set $([0,x]\times[0,y])\setminus([0,x']\times[y',y])$ contained in Φ are erased by erasers from the right.

The following proposition gives the expression of λ_ε and the law of D_ε as a function of the values of G and H.

Proposition 4 *We have:*

(i) $\lambda_\varepsilon = \int_0^\varepsilon G(r,v)^2 f(v)dv$;

(ii) *For every* $t \geq 2r$,

$$\mathbf{P}\{D_\varepsilon \geq t\} = \left(\frac{G(r,\varepsilon)}{\lambda_\varepsilon}\int_0^\varepsilon G(r,v)e^{-rF(v)}f(v)dv\right)\cdot e^{-(t-2r)F(\varepsilon)}; \tag{20}$$

(iii) *For every* $t \in [r, 2r]$,

$$\mathbf{P}\{D_\varepsilon \geq t\} = \frac{1}{\lambda_\varepsilon}\int_0^\varepsilon G(r,v)H(r,\varepsilon,2r-t,v)e^{-F(v)(t-r)}f(v)dv. \tag{21}$$

Proof. (i) According to the point (i) of Lemma 3, it suffices to demonstrate the equality

$$\mathbf{P}\{(0,v) \notin \mathcal{A}(\Phi)\} = \mathbf{P}\{(0,v) \text{ not erased }\} = G(r,v)^2, \quad \forall v \in [0,\varepsilon]. \tag{22}$$

Besides, the point $(0,v)$ is not erased if and only if there is no eraser in $[-r,r]\times[0,v]$, which means if the points of $\Phi\cap([-r,r]\times[0,v])$ have been erased themselves.

In that case, the points of $\Phi\cap([0,r]\times[0,v])$ (respectively of $\Phi\cap([-r,0]\times[0,v])$) could have been erased only by erasers on the right (respectively on the left).

So we have the equivalence

$$(0,v)\notin\mathcal{A}(\Phi) \iff \begin{cases} \Phi^+\cap([0,r]\times[0,v]) & \subset \mathcal{A}(\Phi^+) \\ \Phi^-\cap([-r,0]\times[0,v]) & \subset \mathcal{A}(\Phi^-). \end{cases} \tag{23}$$

Consequently, using the independence of Φ^+ and Φ^-, we obtain

$$\mathbf{P}\{(0,v)\notin\mathcal{A}(\Phi)\} = \mathbf{P}\{\Phi^+\cap([0,r]\times[0,v])\subset\mathcal{A}(\Phi^+)\}\cdot\mathbf{P}\{\Phi^-\cap([-r,0]\times[0,v])\subset\mathcal{A}(\Phi^-)\}.$$

Let v be in $[0,\varepsilon]$. Equality (22) is a direct consequence of:

$$\mathbf{P}\{\Phi^+\cap([0,r]\times[0,v])\subset\mathcal{A}(\Phi^+)\} = \mathbf{P}\{\Phi^-\cap([-r,0]\times[0,v])\subset\mathcal{A}(\Phi^-)\} = G(r,v).$$

(ii)(iii) In order to determine the law of D_ε, from point (ii) of Lemma 3, it is sufficient to calculate the expression

$$\mathbf{P}\{(0,v)\notin\mathcal{A}(\Phi); v(0,\Lambda_{\varepsilon,v})\geq t\}, \quad t\geq r, v\in[0,\varepsilon].$$

We proceed as for (i) and we obtain the equality

$$\{(0,v) \text{ not erased }; v(0,\Lambda_{\varepsilon,v})\geq t\} = A^-\cap A_t^+, \tag{24}$$

where A^- and A_t^+ are two independent events defined by

$$\begin{aligned} A^- &= \{\Phi\cap([-r,0]\times[0,v]) \text{ erased by the left}\}, \\ A_t^+ &= \{\Phi\cap([0,r]\times[0,v]\cup[r,t]\times[0,\varepsilon]) \text{ erased by the right}\}. \end{aligned}$$

Let us remark that

$$\mathbf{P}(A^-) = G(r,v). \tag{25}$$

Consequently we obtain the formula

$$\mathbf{P}\{(0,v)\notin\mathcal{A}(\Phi);v(0,\Lambda_{\varepsilon,v})\geq t\}=G(r,v)\cdot\mathbf{P}(A_t^+). \tag{26}$$

It then remains to determine $\mathbf{P}(A_t^+)$. The computation of this probability depends whether $t\geq 2r$ or $t\in[r,2r]$.

First case: $t\geq 2r$**.**

Since $v\leq\varepsilon$ and a point of Φ can be erased only by an eraser located at a distance smaller than r on the x-axis, we can rewrite the event A_t^+ in the following terms:

$$A_t^+=\{\Phi\cap([0,r]\times[0,v]\cup[r,t-r]\times[0,\varepsilon])=\emptyset\}\cap\{\Phi\cap([t-r,t]\times[0,\varepsilon])\text{ erased by the right}\}, \tag{27}$$

the two events of the intersection being independent.

The Poissonian property of Φ provides the equality

$$\mathbf{P}\{\Phi\cap([0,r]\times[0,v]\cup[r,t-r]\times[0,\varepsilon])=\emptyset\}=e^{-\nu([0,r]\times[0,v]\cup[r,t-r]\times[0,\varepsilon])}=e^{-(t-2r)F(\varepsilon)}e^{-rF(v)}. \tag{28}$$

Φ being invariant under horizontal translations T^{t-r}, we have

$$\mathbf{P}\{\Phi\cap([t-r,t]\times[0,\varepsilon])\text{ erased by the right}\}=G(r,\varepsilon). \tag{29}$$

So we deduce from formulas (27), (28) and (29):

$$\mathbf{P}(A_t^+)=G(r,\varepsilon)e^{-rF(v)}e^{-(t-2r)F(\varepsilon)}.$$

Relation (20) follows immediately.

Second case: $t\in[r,2r]$**.**

We rewrite the event A_t^+ as the intersection of two independent events:

$$A_t^+=\{\Phi\cap([0,t-r]\times[0,v]=\emptyset\}\cap\{\Phi\cap([t-r,t]\times[0,\varepsilon]\setminus[t-r,r]\times[v,\varepsilon])\text{ erased by the right}\}.$$

The invariance under horizontal translations of Φ implies that

$$\mathbf{P}\{\Phi\cap([t-r,t]\times[0,\varepsilon]\setminus[t-r,r]\times[v,\varepsilon])\text{ erased by the right }\}=H(r,\varepsilon,2r-t,v),$$

and we then have the equality

$$\mathbf{P}(A_t^+)=H(r,\varepsilon,2r-t,v)e^{-F(v)(t-r)}.$$

By using (26), we can reach the same conclusion as in the first case.

$\square$

3 Explicit expression of the mean crack number and distribution function of the typical inter-crack distance.

In order to obtain explicitely the mean crack number and the distribution of D_ε (resp. $|I_0^e|$), we have to determine the functions G and H. We prove in subsection 3.1 that G verifies an integral equation. The key point is the explicit calculation of G in subsection 3.2. This allows the function H to be determined. Since λ_ε and the distributions of D_ε (resp. $|I_0^\varepsilon|$) are expressed through G and H, we calculate these quantities. We also consider the typical strain level L_ε for the points of Λ_ε, defining the joint distribution of the couple $(D_\varepsilon, L_\varepsilon)$ in the Palm sense:

$$\mathbf{E}\left\{h(D_\varepsilon, L_\varepsilon)\right\} = \frac{1}{\lambda_\varepsilon |B|}\mathbf{E}\left\{\sum_{(x,y)\in\Psi\cap(B\times[0,\varepsilon])} h(v(x,\Lambda_\varepsilon) - x, y)\right\}, \tag{30}$$

for any measurable $h : (\mathbb{R}_+)^2 \longrightarrow \mathbb{R}_+$ and all fixed Borel set $B \subset \mathbb{R}$.

We prove that $(D_\varepsilon, L_\varepsilon)$ has a density and we determine it.

3.1 A functional equation satisfied by G.

Let us denote

$$G_1(x, y) = G(x, F^{-1}(y)), \quad 0 \le x \le r, y \ge 0.$$

Then G_1 satisfies the following functional equation.

Proposition 5 *For every $0 \le x \le r$, $y \ge 0$,*

$$G_1(x, y) = 1 - e^{-xy}\int_0^x \int_0^y G_1(r-u, y-v)e^{uv}(1+uv)dudv. \tag{31}$$

Proof. Let us recall first that for every fixed $x \in [0, r]$ and $y \in [0, \varepsilon]$, we have the equality in law

$$\Phi^+ \cap ([0, x] \times [0, y]) \overset{law}{=} \{(X_i, Y_i); 1 \le i \le N\},$$

where:

(i') $\{(X_i, Y_i); i \ge 1\}$ is a sequence of independent and identically distributed variables, of density $(1/(xF(y)))\mathbf{1}_{[0,x]}(u)\mathbf{1}_{[0,y]}f(v)$;

(ii') N is a Poisson variable of mean value $\mathbf{E}N = xF(y)$, independent of the preceding sequence.

Let us define for all $n \ge 1$,

$$(M_1^{(n)}, M_2^{(n)}) = (\inf_{1\le i\le n} X_i, \inf_{1\le i\le n} Y_i).$$

It is easily verified that the law of the couple $(M_1^{(n)}, M_2^{(n)})$ is given by

$$\mathbf{P}\{M_1^{(n)} \geq u; M_2^{(n)} \geq v\} \quad = \quad \left(1 - \frac{u}{x}\right)^n \left(1 - \frac{F(v)}{F(y)}\right)^n, \quad u \in [0,x], v \in [0,y]. \quad (32)$$

The key point is the following: let $(M_1^{(n)}, Y_0)$ (respectively $(X_0, M_2^{(n)})$) be the point of Φ, of first coordinate $M_1^{(n)}$ (resp. of second coordinate $M_2^{(n)}$).

The points of $\Phi^+ \cap ([0,x] \times [0,y])$ cannot be erased by more than one eraser (X,Y). Since (X,Y) has to erase $(M_1^{(n)}, Y_0)$ (resp. $(X_0, M_2^{(n)})$), then $X \leq M_1^{(n)} + r$ (resp. $Y \leq M_2^{(n)}$).

Consequently, that happens if and only if:
1. either $N = 0$.
2. or $N = n$, $n \geq 1$, and there is an eraser in $([x, M_1^{(n)} + r] \times [0, M_2^{(n)}])$.

This argument associated with the formula (32) provides the following calculation of G.

$$\begin{aligned}
&G(x,y)\\
&\quad = \mathbf{P}\{\Phi^+ \cap ([0,x] \times [0,y]) \text{ erased by the right}\}\\
&\quad = \mathbf{P}\{N = 0\}\\
&\qquad + \sum_{n\geq 1} \mathbf{P}\{N = n\} \mathbf{P}\{\Phi^+ \cap ([x, M_1^{(n)} + r] \times [0, M_2^{(n)}]) \text{ not totally erased by the right}\}\\
&\quad = e^{-xF(y)} \left[1 + \sum_{n\geq 1} \frac{(xF(y))^n}{n!} \int_0^x \int_0^y (1 - G(u + r - x, v)) \mathbf{P}(M_1^{(n)} \in du, M_2^{(n)} \in dv)\right]\\
&\quad = 1 - e^{-xF(y)} \sum_{n\geq 1} n^2 \frac{(xF(y))^{n-1}}{n!}\\
&\qquad\qquad \cdot \int_0^x \int_0^y G(u + r - x, v) \left(1 - \frac{u}{x}\right)^{n-1} \left(1 - \frac{F(v)}{F(y)}\right)^{n-1} f(v) du dv\\
&\quad = 1 - e^{-xF(y)}\\
&\qquad\qquad \cdot \int_0^x \int_0^y G(u + r - x, v) e^{(x-u)(F(y)-F(v))} (1 + (x-u)(F(y) - F(v))) f(v) du dv.
\end{aligned}$$

Taking the change of variables (in the integral) $u' = x - u$, $v' = F(y) - F(v)$, we deduce relation (31) from the preceding equality.

$\square$

Let us consider the bounded operator L on the space of continuous functions $C([0,r] \times \mathbb{R}_+)$ (endowed with the topology of uniform convergence on any compact set) defined by

$$L(Q) : (x,y) \longmapsto e^{-xy} \int_0^x \int_0^y Q(r - u, y - v) e^{uv} (1 + uv) du dv, \quad x \in [0,r], y \geq 0, \quad (33)$$

where $Q \in C([0,r] \times \mathbb{R}_+)$.

The following theorem provides uniqueness of the solution of the functional equation (31) in the space $C([0,r] \times \mathbb{R}_+)$.

Proposition 6 *We have*

$$G_1 = \sum_{n\geq 0}(-1)^n L^n(1), \tag{34}$$

the convergence of the series being uniform on $[0,r]\times[0,k]$*, for any* $k>0$.

Proof. Equation (31) can be rewritten as

$$G_1 + L(G_1) = 1, \quad G_1 \in C([0,r]\times\mathbb{R}_+). \tag{35}$$

Let us remark that

$$L(1)(x,y) = 1 - e^{-xy}, \quad x\in[0,r], y\in\mathbb{R}_+. \tag{36}$$

Consider $k>0$. We deduce easily that the restriction of L on the space of continuous functions on $[0,r]\times[0,k]$ has an infinite norm equal to $(1-e^{rk})$.

Consequently, if I denotes the identity operator on the space $C([0,r]\times[0,k])$, the series

$$\sum_{n\geq 0}(-1)^n L^n$$

converges normally and is the inverse of $(I+L)$. So we obtain the result (34).

□

The functional equation (31) is the key point to calculate the expressions of the functions G and H. We then will deduce from Proposition 4 the mean crack number and distribution function of the typical inter-crack distance.

3.2 Explicit expression of λ_ε and $\mathbf{P}\{D_\varepsilon \geq t\}$, $t\geq r$.

From now on we denote α the function on $\mathbb{R}_+$ defined by

$$\alpha(s) = \exp\left\{-\int_0^{rs}\frac{1-e^{-t}}{t}dt\right\}, \quad s\geq 0.$$

Let us remark that α satisisfies the two following identities:

$$\begin{aligned}(\alpha(t)t)' &= \alpha(t)e^{-rt}, & (37)\\ \alpha(t) &= \frac{e^{-\gamma}}{rt}\exp\{-\mathrm{Ei}(1,rt)\}, \quad t>0, & (38)\end{aligned}$$

where γ is Euler's constant and $\mathrm{Ei}(n,x) = \int_1^{+\infty}\frac{e^{-xs}}{s}ds$.

Proposition 7 *For* $0\leq x\leq r$, $y\geq 0$,

$$G_1(x,y) = 1 - \int_0^y \alpha(s)\frac{1-e^{-sx}}{s}ds = 1 - \int_0^y \exp\left\{-\int_0^{rs}\frac{1-e^{-t}}{t}dt\right\}\frac{1-e^{-sx}}{s}ds.$$

In particular,

$$G_1(r,y) = \alpha(y) = \exp\left\{-\int_0^{ry}\frac{1-e^{-v}}{v}dv\right\}. \tag{39}$$

Proof. Let us recall that Proposition 6 provides uniqueness of the solution of the integral equation (31) in the space $C([0,r]\times\mathbb{R}_+)$. So it suffices to verify that the continuous function

$$U(x,y) = 1 - V(x,y) = 1 - \int_0^y \alpha(s)\frac{1-e^{-sx}}{s}ds, \quad x\in[0,r], y\geq 0,$$

satisfies the identity $U + L(U) = 1$, or equivalently using (36),

$$L(V)(x,y) = U(x,y) - e^{-xy}. \tag{40}$$

We need to calculate $L(V)$, L being the operator defined by (33).

For $x\in[0,r]$ and $y\geq 0$ fixed, we have

$$\begin{aligned}
L(V)(x,y) &= e^{-xy}\int_0^x\int_0^y\int_0^{y-v}\alpha(s)\frac{1-e^{-s(r-u)}}{s}e^{uv}(1+uv)dudvds\\
&= e^{-xy}\int_0^x\int_0^y\alpha(s)\frac{1-e^{-s(r-u)}}{s}\left[\int_0^{y-s}e^{uv}(1+uv)dv\right]dsdu\\
&= e^{-xy}\int_0^y\frac{\alpha(s)}{s}\left[\int_0^x(y-s)(1-e^{-s(r-u)})e^{u(y-s)}du\right]ds\\
&= \int_0^y\frac{\alpha(s)}{s}\left\{e^{-xs}-e^{-xy}-(y-s)e^{-rs}\frac{1-e^{-xy}}{y}\right\}ds\\
&= -\int_0^y\frac{\alpha(s)}{s}(1-e^{-xs})ds\\
&\qquad -\frac{1-e^{-xy}}{y}\left(\int_0^y(y-s)\alpha(s)\frac{e^{-sr}-1}{s}ds-\int_0^y\alpha(s)ds\right)\\
&= U(x,y)-1-\frac{1-e^{-xy}}{y}\left(\int_0^y(y-s)\alpha'(s)ds-\int_0^y\alpha(s)ds\right)\\
&= U(x,y)-1+(1-e^{-xy}) = U(x,y)-e^{-xy}.
\end{aligned}$$

We then obtain (40), which implies Proposition 7.

□

Let us define the process of the cracks on the positive half-line in the following way: we construct an intermediary process Ψ^+ on $(\mathbb{R}_+)^2$ with the same method as at the beginning of the first section for Ψ, replacing the two-dimensional process Φ by its intersection Φ^+ on $(\mathbb{R}_+)^2$. Then Λ_ε^+ is a point process on $\mathbb{R}_+$, given by the equality

$$\Lambda_\varepsilon^+ = \{x\in\mathbb{R}_+; \exists y\in[0,\varepsilon] \mid (x,y)\in\Psi^+\}. \tag{41}$$

Let us consider the first positive crack position at the strain level ε,

$$X_1^\varepsilon = \inf\Lambda_\varepsilon^+. \tag{42}$$

The calculation of the law of X_1^ε, $\varepsilon\geq 0$, is essential to obtain the explicit expression of the function $H(r,\cdot,\cdot,\cdot)$ defined in (19). The following theorem provides the exact distribution of X_1^ε.

Theorem 8 *The law of X_1^ε is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}$, with density $\varphi_{X_1^\varepsilon}$ such that*

$$\varphi_{X_1^\varepsilon}(x) = \begin{cases} F(\varepsilon)\alpha(F(\varepsilon))e^{-F(\varepsilon)(x-r)} & \text{if } x \geq r \\ \int_0^{F(\varepsilon)} \alpha(v)e^{-xv}dv & \text{if } x \in [0,r]. \end{cases} \tag{43}$$

Proof. Using Proposition 7, it suffices to verify that

$$\mathbf{P}\{X_1^\varepsilon \geq x\} = \begin{cases} G(x,\varepsilon) & \text{if } x \in [0,r] \\ e^{-(x-r)F(\varepsilon)}\alpha(F(\varepsilon)) & \text{if } x \geq r. \end{cases} \tag{44}$$

Let us notice the equality of events

$$\{\Phi^+ \cap ([0,x]\times[0,\varepsilon]) \text{ erased by the right }\} = \{X_1^\varepsilon \geq x\}. \tag{45}$$

The formula (44) for $x \in [0,r]$ follows directly from (45) and (18).

When $x \geq r$, using (45) and the invariance by any horizontal translation of Φ^+, we have

$$\begin{aligned} &\mathbf{P}\{X_1^\varepsilon \geq x\} \\ &\quad = \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,\varepsilon]) \text{ erased by the right }\} \\ &\quad = \mathbf{P}\{(\Phi^+ \cap ([0,x-r]\times[0,\varepsilon]) = \emptyset) \cap (\Phi^+ \cap ([x-r,x]\times[0,\varepsilon]) \text{ erased by the right })\} \\ &\quad = \mathbf{P}\{\Phi^+ \cap ([0,x-r]\times[0,\varepsilon]) = \emptyset\}\cdot\mathbf{P}\{\Phi^+ \cap ([0,r]\times[0,\varepsilon]) \text{ erased by the right })\} \\ &\quad = e^{-(x-r)F(\varepsilon)}G(r,\varepsilon). \end{aligned}$$

This proves the second part of (44). In particular,

$$\mathbf{P}\{\Phi^+ \cap ([0,r]\times[0,\varepsilon]) \text{ erased by the right }\} = \mathbf{P}\{X_1^\varepsilon \geq r\} = \alpha(F(\varepsilon)). \tag{46}$$

□

Proposition 9 *For every $0 \leq x \leq r$, $0 \leq y \leq \varepsilon$,*

$$H(r,\varepsilon,x,y) \quad = \quad \alpha(F(y)) - e^{-xF(y)}\int_{F(y)}^{F(\varepsilon)} \alpha(s)\frac{1-e^{-(r-x)s}}{s}ds.$$

Proof. Let us remark the equality of events

$$\{\Phi^+ \cap ([0,r]\times[0,\varepsilon] \setminus [0,x]\times[y,\varepsilon]) \text{ erased by the right }\} = A_1 \cup A_2,$$

where

$$\begin{aligned} A_1 &= \{X_1^y \in [r, x+r]\}, \\ A_2 &= \{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset; \Phi^+ \cap ([x,r]\times[0,\varepsilon]) \text{ erased by the right}\}. \end{aligned}$$

So we obtain the following formula which is the key point of the proof of Theorem 9:

$$H(r,\varepsilon,x,y) = \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_1 \cap A_2). \tag{47}$$

Using Theorem 8, we have

$$\mathbf{P}(A_1) = \int_r^{x+r} F(y)\alpha(F(y))e^{-F(y)(u-r)}du = \alpha(F(y))(1-e^{-xF(y)}). \tag{48}$$

Moreover, the invariance of Φ^+ by any positive translation and Theorem 7 imply that

$$\begin{aligned}
\mathbf{P}(A_2) &= \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \\
&\qquad \cdot \mathbf{P}\{\Phi^+ \cap ([0,r-x]\times[0,\varepsilon]) \text{ erased by the right}\} \\
&= e^{-xF(y)}G(r-x,\varepsilon) \\
&= e^{-xF(y)}\left(1-\int_0^{F(\varepsilon)} \alpha(s)\frac{1-e^{-s(r-x)}}{s}ds\right).
\end{aligned} \tag{49}$$

It remains to determine $\mathbf{P}(A_1 \cap A_2)$. To this end, we remark that the law of the process Φ^+ conditioned by the event $\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\}$ is the same as $T^x(\Phi^+)$.

Consequently,

$$\begin{aligned}
\mathbf{P}(A_1 \cap A_2) &= \mathbf{P}\{X_1^y \in [r, x+r] | \Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \\
&\qquad \cdot \mathbf{P}\{\Phi^+ \cap ([0,x]\times[0,y]) = \emptyset\} \\
&= \mathbf{P}\{X_1^y \in [r-x, r]\} \cdot e^{-xF(y)} \\
&= e^{-xF(y)}(\mathbf{P}\{X_1^y \geq r-x\} - \mathbf{P}\{X_1^y > r\}) \\
&= e^{-xF(y)} \int_0^{F(y)} \alpha(v)\frac{e^{-(r-x)v}-e^{-rv}}{v}dv.
\end{aligned} \tag{50}$$

Inserting formulae (48), (49) and (50) in (47), we get

$$\begin{aligned}
H(r,\varepsilon,x,y) &= \alpha(F(y))(1-e^{-xF(y)}) + e^{-xF(y)}\left\{1-\int_0^{F(\varepsilon)}\alpha(s)\frac{1-e^{-s(r-x)}}{s}ds\right. \\
&\qquad \left. -\int_0^{F(y)}\alpha(s)\frac{e^{-s(r-x)}-e^{-rs}}{s}ds\right\}.
\end{aligned} \tag{51}$$

We calculate the last integral in the following way:

$$\begin{aligned}
\int_0^{F(y)}\alpha(s)\frac{e^{-s(r-x)}-e^{-rs}}{s}ds &= \int_0^{F(y)}\alpha(s)\frac{e^{-s(r-x)}-1}{s}ds + \int_0^{F(y)}\alpha(s)\frac{1-e^{-rs}}{s}ds \\
&= \int_0^{F(\varepsilon)}\alpha(s)\frac{e^{-s(r-x)}-1}{s}ds - \int_{F(y)}^{F(\varepsilon)}\alpha(s)\frac{e^{-s(r-x)}-1}{s}ds \\
&\qquad - [\alpha(s)]_0^{F(y)} \\
&= \int_{F(y)}^{F(\varepsilon)}\alpha(s)\frac{1-e^{-s(r-x)}}{s}ds - \int_0^{F(\varepsilon)}\alpha(s)\frac{1-e^{-s(r-x)}}{s}ds \\
&\qquad +1-\alpha(F(y)).
\end{aligned} \tag{52}$$

Combining the equalities (51) et (52), we obtain Proposition 9.

$\square$

Returning to Proposition 4, we get from Propositions 7 and 9 the following expressions for the mean crack number and distribution function of the inter-crack distance.

Theorem 10 *We have*

(i) $\lambda_\varepsilon = \int_0^{F(\varepsilon)} \alpha(v)^2 dv$;

(ii) D_ε *has a density* $\varphi_{D\varepsilon}$ *with respect to the Lebesgue measure on* $[r, +\infty)$ *such that*

$$\varphi_{D_\varepsilon}(x) = \begin{cases} \frac{F(\varepsilon)^2}{\lambda_\varepsilon}\alpha(F(\varepsilon))^2 e^{-(x-2r)F(\varepsilon)} & \textit{if } x > 2r \\ \frac{2}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} e^{-(x-r)v}\alpha(v)^2 v dv & \textit{if } r \le x \le 2r. \end{cases}$$

Remark 11 (i) The points (i)-(ii) of Theorem 10 were previously obtained in a very different form through heuristic methods by Widom (see [19], page 3893, results (37)-(39)).
(ii) Let us remark that the distribution of D_ε has a decreasing density on $[r, +\infty)$, with a transition at $2r$. Actually, in the interval $[2r, +\infty)$, the law is exponential which means that D_ε has all its moments finite. Applying Proposition 2, we obtain that for all $n \ge 1$,

$$\frac{1}{N_L}\sum_{I \in \mathcal{I}_L} |I|^n \longrightarrow \mathbf{E}(D_\varepsilon^n), \qquad \textit{when } L \to +\infty.$$

Besides, it is easy to verify that the first moment of D_ε satisfies the equality (13), as we already noticed.

Proof. The point (i) follows immediately from Proposition 7.
To prove the point (ii), it suffices to demonstrate that

$$\mathbf{P}\{D_\varepsilon \ge t\} = \begin{cases} \frac{F(\varepsilon)}{\lambda_\varepsilon}\alpha(F(\varepsilon))^2 e^{-(t-2r)F(\varepsilon)}; & \text{if } t \ge 2r \\ \frac{2}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)^2 e^{-(t-r)v} dv - 1 & \text{if } t \in [r, 2r]. \end{cases} \tag{53}$$

Due to Proposition 4 and (37), we have when $t > 2r$,

$$\begin{aligned} \mathbf{P}\{D_\varepsilon \ge t\} &= \left(\frac{\alpha(F(\varepsilon))}{\lambda_\varepsilon}\int_0^\varepsilon \alpha(F(v))e^{-rF(v)}f(v)dv\right) \cdot e^{-(t-2r)F(\varepsilon)} \\ &= \frac{\alpha(F(\varepsilon))}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)e^{-rv}dv \cdot e^{-(t-2r)F(\varepsilon)} \\ &= \frac{\alpha(F(\varepsilon))}{\lambda_\varepsilon}[v\alpha(v)]_0^{F(\varepsilon)} \cdot e^{-(t-2r)F(\varepsilon)} \\ &= \frac{F(\varepsilon)}{\lambda_\varepsilon}\alpha(F(\varepsilon))^2 e^{-(t-2r)F(\varepsilon)}. \end{aligned}$$

Let us now focus on the case where $t \in [r, 2r]$. Using Propositions 7 and 9, we obtain:

$$\begin{aligned}
\mathbf{P}\{D_\varepsilon \geq t\} &= \frac{1}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)e^{-(t-r)v}\left\{\alpha(v) - e^{(2r-t)v}\int_v^{F(\varepsilon)} \alpha(s)\frac{1-e^{-(t-r)s}}{s}ds\right\}dv \\
&= \frac{1}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)^2 e^{-(t-r)v}dv \\
&\qquad -\frac{1}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(s)\frac{1-e^{-(t-r)s}}{s}\left[\int_0^s e^{-rv}\alpha(v)dv\right]ds \\
&= \frac{1}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)^2 e^{-(t-r)v}dv - \frac{1}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(s)\frac{1-e^{-(t-r)s}}{s}s\alpha(s)ds \\
&= \frac{2}{\lambda_\varepsilon}\int_0^{F(\varepsilon)} \alpha(v)^2 e^{-(t-r)v}dv - 1.
\end{aligned}$$

This completes the proof of the equality (53) and consequently of Theorem 10.

□

A direct consequence of Theorem 10 is the explicit distribution of $|I_0^\varepsilon|$.

Proposition 12 *The law of $|I_0^\varepsilon|$ has a density $\varphi_{|I_0^\varepsilon|}$ on $[r, +\infty)$ such that*

$$\varphi_{|I_0^\varepsilon|}(x) = \begin{cases} F(\varepsilon)^2\alpha(F(\varepsilon))^2 x e^{-(x-2r)F(\varepsilon)} & \text{if } x > 2r \\ 2x\int_0^{F(\varepsilon)} e^{-(x-r)v}\alpha(v)^2 v dv & \text{if } r \leq x \leq 2r \end{cases}$$

Proof. Relation (13) and equality (14) applied to $h(x) = x$ give that

$$\mathbf{E}\left(\frac{1}{|I_0^\varepsilon|}\right) = \lambda_\varepsilon.$$

We conclude easily by inserting the result (ii) of Theorem 10 in (14).

□

We conclude this section by generalizing point (ii) of Theorem 10: we determine the joint density of the couple $(D_\varepsilon, L_\varepsilon)$ (defined in the Palm sense by (30)).

Theorem 13 *We have:*
(i) L_ε has a density φ_{L_ε} such that

$$\varphi_{L_\varepsilon}(y) = \frac{1}{\lambda_\varepsilon}\alpha(F(y))^2 f(y)\mathbf{1}_{[0,\varepsilon]}(y); \tag{54}$$

(ii) the law of D_ε conditionally to L_ε has a density $\Pi^{D_\varepsilon}(L_\varepsilon; \cdot)$ such that for every $y \in [0, \varepsilon], u \geq 0$

$$\begin{aligned}
\Pi^{D_\varepsilon}(y; u) = \mathbf{1}_{\{u>2r\}}&\frac{F(\varepsilon)\alpha(F(\varepsilon))}{\alpha(F(y))}e^{-rF(y)}e^{-(u-2r)F(\varepsilon)} \\
&+\mathbf{1}_{\{r\leq u\leq 2r\}}\left\{F(y)e^{-(u-r)F(y)} + \frac{e^{-rF(y)}}{\alpha(F(y))}\int_{F(y)}^{F(\varepsilon)} e^{-(u-r)v}\alpha(v)dv\right\}.
\end{aligned}\tag{55}$$

Proof. Fixing $t \geq r, s \in [0, \varepsilon]$, we use the definition (30) of $(D_\varepsilon, L_\varepsilon)$ and apply Slivnyak's formula (17) as in the proof of Lemma 3 to obtain that

$$\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} = \frac{1}{\lambda_\varepsilon} \int_0^s \mathbf{P}\{(0, v) \notin \mathcal{A}(\Phi); v(0, \Lambda_{\varepsilon,v}) \geq t\} f(v) dv.$$

Consequently, we get as in Proposition 4 that

$$\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} = \begin{cases} \left(\frac{G(r,\varepsilon)}{\lambda_\varepsilon} \int_0^s G(r, v) e^{-rF(v)} f(v) dv \right) \cdot e^{-(t-2r)F(\varepsilon)} & \text{if } t \geq 2r \\ \frac{1}{\lambda_\varepsilon} \int_0^s G(r, v) H(r, \varepsilon, 2r - t, v) e^{-F(v)(t-r)} f(v) dv & \text{if } t \in [r, 2r]. \end{cases} \tag{56}$$

It then suffices to insert the expressions of G and H (cf Propositions 7 and 9) into (56) to deduce that

$$\begin{aligned} &\mathbf{P}\{D_\varepsilon \geq t; L_\varepsilon \leq s\} \\ &= \begin{cases} \frac{\alpha(F(\varepsilon))}{\lambda_\varepsilon} F(s) \alpha(F(s)) e^{-(t-2r)F(\varepsilon)} & \text{if } t \geq 2r \\ \frac{1}{\lambda_\varepsilon} \int_0^{F(s)} \alpha(v)^2 (2e^{-(t-r)v} - 1) dv & \\ \qquad\qquad - \frac{F(s)\alpha(F(s))}{\lambda_\varepsilon} \int_{F(s)}^{F(\varepsilon)} \alpha(v) \frac{1 - e^{-(t-r)v}}{v} dv & \text{if } t \in [r, 2r]. \end{cases} \end{aligned}$$

Points (i) and (ii) of Theorem 13 are easy consequences of this last equality.

□

4 The law of the successive cracks on the half-line.

We got a good understanding of Λ_ε through Palm interpretation. Another description of Λ_ε is possible by considering the one-sided process Λ_ε^+ defined in (41) on the positive half-line. We fix arbitrarily an origin and we enumerate the points of Λ_ε^+ as follows.

$$\Lambda_\varepsilon^+ = \{X_n^\varepsilon; n \in \mathbb{N}^*\},$$

where $0 < X_1^\varepsilon < X_2^\varepsilon < \cdots < X_n^\varepsilon < \cdots$, $n \geq 1$.
Let Y_n^ε be the positive real number such that

$$(X_n^\varepsilon, Y_n^\varepsilon) \in \Psi^+ \cap (\mathbb{R}_+ \times [0, \varepsilon]).$$

In the preceding section, we determined in Theorem 8 the distribution of X_1^ε. The aim of this section is the description of the distribution of $\{(X_i^\varepsilon, Y_i^\varepsilon); 1 \leq i \leq n\}$ for any $n \geq 1$. A first answer is given by a recursive algorithm (cf Theorems 14 and 15): we compute the distribution of $(X_1^\varepsilon, Y_1^\varepsilon)$ and the distribution of $(X_{n+1}^\varepsilon, Y_{n+1}^\varepsilon)$ conditionally to $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$. We interpret this result using a Markov chain model (cf Theorem 16) and we prove the convergence in law of the couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$ to $(D_\varepsilon, L_\varepsilon)$ (cf Theorem 17).

We observe in particular that $\{X_n^\varepsilon; n \geq 1\}$ is not a renewal sequence, for instance $(X_2^\varepsilon - X_1^\varepsilon)$ is not independent of X_1^ε. However we prove (cf Theorem 18) that $\{X_n^\varepsilon; n \geq 1\}$ is a conditioned renewal process.

We start with the density of $(X_1^\varepsilon, Y_1^\varepsilon)$.

Theorem 14 *The law of the couple $(X_1^\varepsilon, Y_1^\varepsilon)$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^2$, with density $\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}$ such that for every $u, v \in \mathbb{R}$,*

$$\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}(u,v) = \left(\mathbf{1}_{\{u>r\}} e^{-(u-r)F(\varepsilon)} e^{-rF(v)} + \mathbf{1}_{\{0\le u\le r\}} e^{-uF(v)}\right) \alpha(F(v)) f(v) \mathbf{1}_{\{0\le v\le \varepsilon\}}. \quad (57)$$

Proof. Let $x \geq 0$ and $0 \leq y \leq \varepsilon$. It suffices to prove that:

$$\mathbf{P}\{X_1^\varepsilon \geq x; Y_1^\varepsilon \leq y\} = \begin{cases} \frac{F(y)}{F(\varepsilon)}\alpha(F(y))e^{-(x-r)F(\varepsilon)} & \text{if } x > r \\ 1-\alpha(F(y)) & \\ \qquad -\int_0^{F(y)} \frac{\alpha(v)}{v}(1-e^{-xv})dv + \frac{F(y)}{F(\varepsilon)}\alpha(F(y)) & \text{otherwise.} \end{cases} \quad (58)$$

We notice that

$$\begin{aligned} \mathbf{P}\{X_1^\varepsilon \geq x; Y_1^\varepsilon \leq y\} &= \mathbf{P}\{X_1^\varepsilon \geq x; X_1^\varepsilon = X_1^y\} \\ &= \mathbf{P}\{X_1^y \geq x; X_1^\varepsilon = X_1^y\} \\ &= \int_x^{+\infty} \mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} \mathbf{P}\{X_1^y \in du\}. \end{aligned} \quad (59)$$

Moreover, $\{X_1^\varepsilon = X_1^y\}$ if and only if there is no positive eraser in $([0, X_1^y] \times [y, \varepsilon])$, which means that for every $u \geq 0$,

$$\mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} = \begin{cases} 1 & \text{if } u \leq r \\ \mathbf{P}\{\Phi \cap [0, u-r] \times [y, \varepsilon] = \emptyset\} & \text{otherwise.} \end{cases}$$

Since Φ is a Poisson point process,

$$\mathbf{P}\{X_1^\varepsilon = X_1^y | X_1^y = u\} = \begin{cases} 1 & \text{if } u \leq r \\ e^{-(u-r)(F(\varepsilon)-F(y))} & \text{otherwise.} \end{cases} \quad (60)$$

Inserting equalities (43) and (60) in (59), we get the result (58), via (52).

□

The following theorem provides the law of the couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_{n+1}^\varepsilon)$ conditionally to $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$.

Proposition 15 *Let $n \geq 1$. Conditionally to $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$, the couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_{n+1}^e)$ has a density $\theta^\varepsilon(Y_n^\varepsilon; \cdot)$, where for every $y \in [0, \varepsilon]$:*

$$\begin{aligned} \theta^\varepsilon(y; u, v) = \Big[& \mathbf{1}_{\{u>2r\}} \mathbf{1}_{\{0\le v\le\varepsilon\}} e^{-r(F(y)+F(v))} e^{-(u-2r)F(\varepsilon)} \\ & + \mathbf{1}_{\{r\le u\le 2r\}} \left\{ \mathbf{1}_{\{0\le v\le y\}} e^{-(u-r)F(y)} e^{-rF(v)} \right. \\ & \left. + \mathbf{1}_{\{y<v\le\varepsilon\}} e^{-rF(y)} e^{-(u-r)F(v)} \right\} \Big] \frac{\alpha(F(v))}{\alpha(F(y))} f(v). \end{aligned}$$

Proof. We set $Z_n = (X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ and $z = (x_1, \cdots, x_{n-1}, x, y_1, \cdots, y_{n-1}, y)$ where $0 < x_1 < \cdots < x_{n-1} < x$ and $y_1, \cdots, y_{n-1}, y$ belong to $[0, \varepsilon]$. It suffices to demonstrate

$$\begin{aligned}
&\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Z_n = z\} \\
&= \begin{cases}
\frac{e^{-rF(y)}}{\alpha(F(y))}\frac{F(v)}{F(\varepsilon)}\alpha(F(v))e^{-(u-2r)F(\varepsilon)} & \text{if } u > 2r \\
(e^{-(u-r)F(y)} - e^{-rF(y)})\frac{\alpha(F(v))F(v)}{\alpha(F(y))F(y)} + e^{-rF(y)}\frac{\alpha(F(v))F(v)}{\alpha(F(y))F(\varepsilon)} & \text{if } r \leq u \leq 2r \\
& \text{and } v \leq y, \\
e^{-(u-r)F(y)} & \\
\quad -\frac{e^{-rF(y)}}{\alpha(F(y))}\left\{\int_{F(y)}^{F(v)}(1-e^{-(u-r)s})\frac{\alpha(s)}{s}ds + \alpha(F(v))\left(1 - \frac{F(v)}{F(\varepsilon)}\right)\right\} & \text{if } r \leq u \leq 2r \\
& \text{and } v > y.
\end{cases}
\end{aligned} \tag{61}$$

Our approach is based on the two properties:
(i) conditionally to Z_n, the law of $\widehat{\Phi} = T^{-X_n^\varepsilon}(\Phi_+) \cap (\mathbb{R}_+)^2$ is the same as Φ_+ conditionally to $\{\Phi_+ \cap ([0, r] \times [0, Y_n^\varepsilon])$ erased by the right$\}$.
(ii) $(X_{n+1}^\varepsilon - X_n^\varepsilon - r, Y_{n+1}^\varepsilon)$ is the first point on the right of the point process $T^{-r}(\widehat{\Phi}) \cap (\mathbb{R}_+)^2$.
Using this remark, (39) and (46), we get

$$\begin{aligned}
&\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Z_n = z\} \\
&= \frac{\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ erased by the right}; X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\}}{\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ erased by the right}\}} \\
&= \frac{1}{\alpha(F(y))}\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ erased by the right}; \\
&\qquad\qquad X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\},
\end{aligned} \tag{62}$$

where $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ is the first point on the right of the process $T^{-r}(\Phi_+) \cap (\mathbb{R}_+)^2$, distributed as $(X_1^\varepsilon, Y_1^\varepsilon)$.

First case: $u > 2r$.

We then have

$$\begin{aligned}
&\{\Phi_+ \cap ([0, r] \times [0, y]) \text{ erased by the right}; X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\} \\
&\quad = \{\Phi_+ \cap ([0, r] \times [0, y]) = \emptyset\} \cap \{X_1^\varepsilon(T^{-r}(\Phi_+)) \geq u - r; Y_1^\varepsilon(T^{-r}(\Phi_+)) \leq v\},
\end{aligned}$$

the two events of the intersection being independent.

Using this remark, (62) and the law of $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ given by (58), we obtain

$$\begin{aligned}
\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Z_n = z\}& \\
&= \frac{\mathbf{P}\{\Phi_+ \cap ([0, r] \times [0, y]) = \emptyset\} \cdot \mathbf{P}\{X_1^\varepsilon \geq u - r; Y_1^\varepsilon \leq v\}}{\alpha(F(y))} \\
&= \frac{e^{-rF(y)}}{\alpha(F(y))}\frac{F(v)}{F(\varepsilon)}\alpha(F(v))e^{-(u-2r)F(\varepsilon)}.
\end{aligned} \tag{63}$$

Second case: $r \leq u \leq 2r$.

The independence property is not valid anymore, but $(X_1^\varepsilon(T^{-r}(\Phi_+)), Y_1^\varepsilon(T^{-r}(\Phi_+)))$ is still distributed with density $\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}$ given by (57).

Going back to (62), we get

$$\begin{aligned}\mathbf{P}\{X_{n+1}^\varepsilon - X_n^\varepsilon \geq u; Y_{n+1}^\varepsilon \leq v | Y_n = y\} \\ = \frac{1}{\alpha(F(y))}\int_{u-r}^{+\infty} dw \int_0^v A(w,t,y)\varphi_{(X_1^\varepsilon,Y_1^\varepsilon)}(w,t)dt, \qquad (64)\end{aligned}$$

where for every $w \geq 0, 0 \leq y, t \leq \varepsilon$,

$$\begin{aligned}&A(w,t,y)\\ &= \mathbf{P}\{\Phi_+ \cap ([0,r]\times[0,y]) \text{ erased by the right } | X_1^\varepsilon(T^{-r}(\Phi_+)) = w, Y_1^\varepsilon(T^{-r}(\Phi_+)) = t\}.\end{aligned}$$

It remains to determine the function A. To this end, let us notice that $([0,r]\times[0,y])$ has a non-empty intersection with the domain of relaxation $R(w+r,t)$ if and only if $w \leq r$ and $t \leq y$. Consequently, we obtain

$$A(w,t,y) \;=\; \begin{cases} e^{-wF(y)-(r-w)F(t)} & \text{if } w \leq r \text{ and } t \leq y \\ e^{-rF(y)} & \text{otherwise.}\end{cases} \qquad (65)$$

Inserting formulas (65) and (57) in (64), we deduce the result (61), via (37), which completes the proof of Theorem 15.

□

We explicit the distribution of $\{(X_i^\varepsilon, Y_i^\varepsilon); 1 \leq i \leq n\}$ starting with the law of $\{Y_i^\varepsilon; i \geq 1\}$.

Theorem 16 *$(Y_n^\varepsilon)_{n\geq 1}$ is a homogeneous Markov chain such that:*

(i) Y_1^ε has a density $\varphi_{Y_1^\varepsilon}$ such that

$$\varphi_{Y_1^\varepsilon}(y) = \left(\frac{e^{-rF(y)}}{F(\varepsilon)} + \frac{1-e^{-rF(y)}}{F(y)}\right)\alpha(F(y))f(y)\mathbf{1}_{[0,\varepsilon]}(y);$$

(ii) the transition kernel of $\{Y_n^\varepsilon; n \geq 1\}$ admits a transition probability density $\Pi^{Y,\varepsilon}(y;\cdot)$ such that for every $y, v \in [0,\varepsilon]$,

$$\begin{aligned}\Pi^{Y,\varepsilon}(y;v) = \Bigg(&\mathbf{1}_{\{0\leq v\leq\varepsilon\}}\frac{e^{-r(F(y)+F(v))}}{F(\varepsilon)} + \mathbf{1}_{\{0\leq v\leq y\}}e^{-rF(v)}\frac{1-e^{-rF(y)}}{F(y)}\\ &+\mathbf{1}_{\{y<v\leq\varepsilon\}}e^{-rF(y)}\frac{1-e^{-rF(v)}}{F(v)}\Bigg)\frac{\alpha(F(v))}{\alpha(F(y))}f(v); \qquad (66)\end{aligned}$$

(iii) the stationary law of $(Y_n^\varepsilon)_{n\geq 1}$ is the distribution of L_ε (cf (54));

(iv) conditionally to $(X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ the r.v. $(X_{n+1}^\varepsilon - X_n^\varepsilon)$ has a density which depends only on Y_n^ε and is equal to $\Pi^{D_\varepsilon}(Y_n^\varepsilon;\cdot)$ (recall that $\Pi^{D_\varepsilon}(y;\cdot)$ is given by (55)).

Proof. The points (i), (ii) and (iv) follow easily from Theorems 14 and 15. A straightforward calculation shows that for every $v \in [0, \varepsilon]$,

$$\alpha(F(v))^2 f(v) = \int_0^\varepsilon \Pi^{Y,\varepsilon}(y; v)\alpha(F(y))^2 f(y)dy,$$

which implies the point (iii).

$\square$

Due to Theorem 15, $(X_n^\varepsilon, Y_n^\varepsilon)_{n\geq 1}$ is a Markov chain. It seems natural to investigate its limit distribution. More precisely, we have the following result.

Theorem 17 *The couple $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$ converges in law when $n \to +\infty$, and the limit distribution coincides with the law of $(D_\varepsilon, L_\varepsilon)$ (cf Theorem 13).*

Proof. Let us begin by proving the convergence of the Markov chain $(Y_n^\varepsilon)_{n\geq 1}$ to its stationary distribution μ, the law of L_ε. The transition probability of $(Y_n^\varepsilon)_{n\geq 1}$ has a density $\Pi^{Y,\varepsilon}(y; \cdot)$,
$y \in [0, \varepsilon]$ (see point (ii) of Theorem 16) such that the function $(y, v) \longmapsto \Pi^{Y,\varepsilon}(y; v)$ is continuous and everywhere positive on $(0, \varepsilon]^2$. Consequently, following [5] (example 6.2. of the fifth section), we deduce that $(Y_n^\varepsilon)_{n\geq 1}$ is a Harris chain. Moreover, as we proved the existence of a stationary distribution (see point (iii) of Theorem 16), it is also recurrent (see [5], exercise 6.11. of section 5). Due to the beginning of the section 5.6.c of [5], $(Y_n^\varepsilon)_{n\geq 1}$ is a aperiodic recurrent Harris chain. Consequently, applying the theorem of convergence of Harris chains (see [5], Theorem (6.8)), we deduce that $(Y_n^\varepsilon)_{n\geq 1}$ converges to μ in the sense of the total variation distance $||\cdot||$ (let us notice that the note following Durrett's theorem garantees that the starting law of $(Y_n^\varepsilon)_{n\geq 1}$ given by the point (i) of Theorem 16 satisfies the required hypothesis for the convergence). We recall that the variation distance between two probability measures μ_1, μ_2 with support in $[0, \varepsilon]$ is:

$$||\mu_1 - \mu_2|| = \sup_f \left| \int f d\mu_1 - \int f d\mu_2 \right|,$$

where f belongs to the set of measurable functions defined on $[0, \varepsilon]$, with values in $[0, 1]$. In particular, $(Y_n^\varepsilon)_{n\geq 1}$ converges in distribution to μ.

We now prove the convergence in distribution of $(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)$. Let us consider a continuous bounded measurable function $h : \mathbb{R}_+ \times [0, \varepsilon] \longrightarrow \mathbb{R}$. Using the point (iv) of Theorem 16, we get for every $n \geq 1$,

$$\mathbf{E}\{h(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)\} = \int_0^\varepsilon \left[\int_0^{+\infty} h(x, y)\Pi^{D_\varepsilon}(y; x)dx \right] \mathbf{P}\{Y_n^\varepsilon \in dy\} \tag{67}$$

But $y \longmapsto \int_0^{+\infty} h(x, y)\Pi^{D_\varepsilon}(y; x)dx$ is a bounded and continuous function, therefore

$$\lim_{n\to+\infty} \mathbf{E}\{h(X_{n+1}^\varepsilon - X_n^\varepsilon, Y_n^\varepsilon)\} = \int_0^\varepsilon \left[\int_0^{+\infty} h(x, y)\Pi^{D_\varepsilon}(y; x)dx \right] \mathbf{P}\{L_\varepsilon \in dy\}. \tag{68}$$

Due to Theorem 13, the right hand-side of (68) is equal to $\mathbf{E}\{h(D_\varepsilon, L_\varepsilon)\}$.

□

Let us introduce three probability densities on $\mathbb{R}_+$:

$$\begin{aligned}
\varphi_\eta(x) &= \frac{F(\varepsilon)}{rF(\varepsilon)+1}(\mathbf{1}_{[0,r]}(x)+\mathbf{1}_{(r,+\infty)}(x)e^{-(x-r)F(\varepsilon)}),\\
\varphi_\rho(x) &= \frac{1}{F(\varepsilon)}\mathbf{1}_{[0,\varepsilon]}(x)f(x),\\
\varphi_{\rho'}(x) &= \frac{1}{\int_0^{F(\varepsilon)}\alpha(u)du}\mathbf{1}_{[0,\varepsilon]}(x)\alpha(F(x))f(x).
\end{aligned}$$

In the next theorem, we prove that the process $\{X_n^\varepsilon; n \geq 1\}$ is a conditioned renewal process. In particular, it is not a renewal process.

Theorem 18 *Let $\{A_i; i \geq 1\}$, $\{\eta_i; i \geq 1\}$, $\{\rho_i; i \geq 1\}$ and $\{\rho'_i; i \geq 1\}$ be four mutually independent sequences of i.i.d. variables with densities exponential, φ_η, φ_ρ and $\varphi_{\rho'}$ respectively. Consider also the events*

$$\begin{aligned}
B_n &= \{A_n \geq (\eta_n \wedge r)(F(\rho_n) \vee F(\rho_{n-1})) + r(F(\rho_n) \wedge F(\rho_{n-1}))\},\\
B'_n &= \{A_n \geq (\eta_n \wedge r)(F(\rho'_n) \vee F(\rho_{n-1})) + r(F(\rho'_n) \wedge F(\rho_{n-1}))\},
\end{aligned}$$

with the convention $\rho_0 = 0$ a.s.. Then
(i) The law of the vector $(X_1^\varepsilon, Y_1^\varepsilon)$ is the law of (η_1, ρ'_1) conditioned by B'_1;
(ii) For every $n \geq 2$, the law of the vector

$$(X_1^\varepsilon, Y_1^\varepsilon, X_2^\varepsilon - X_1^\varepsilon, Y_2^\varepsilon, \cdots, X_n^\varepsilon - X_{n-1}^\varepsilon, Y_n^\varepsilon)$$

is the distribution of $(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_{n-1}, \rho_{n-1}, r+\eta_n, \rho'_n)$ conditioned by the event
$B = \cap_{i=1}^{n-1} B_i \cap B'_n$.

Remark 19 We set $Z_n = (X_1^\varepsilon, Y_1^\varepsilon, \cdots, X_n^\varepsilon, Y_n^\varepsilon)$ and $Z'_n = (\eta_1, \rho_1, \cdots, (n-1)r + \eta_1 + \cdots + \eta_n, \rho_n)$. Point (ii) of Theorem 18 implies

$$\varphi_{Z_n} = \frac{1}{\mathbf{P}(B)} e^{-\Gamma_n} \varphi_{Z'_n}, \tag{69}$$

where φ_{Z_n} (resp. $\varphi_{Z'_n}$) denotes the density function of Z_n (resp. Z'_n) and Γ_n is a positive function. In particular, $\varphi_{Z_n} \leq \frac{1}{\mathbf{P}(B)}\varphi_{Z'_n}$.

We may apply the Hit or Miss Monte-Carlo Method (cf. chapter 4 of [18]). We first simulate $Z'_n = \xi$ and we keep it with probability $p = e^{-\Gamma_n(\xi)}$. Otherwise we simulate a new independent copy of Z'_n and so on.

It turns out that this recursive procedure gives a new erasing algorithm based on the renewal sequence $(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_{n-1}, \rho_{n-1}, r+\eta_n, \rho'_n)$. After a calculation, we obtain that

$$\Gamma_n(x_1, y_1, \cdots, x_n y_n) = \nu(\cup_{i=1}^n \mathcal{D}(x_i, y_i)),$$

where

$$\begin{cases} \mathcal{D}(x_1, y_1) = ([(x_1 - r) \vee 0, x_1 + r] \times [0, y_1]) \setminus R(x_2, y_2) \\ \mathcal{D}(x_i, y_i) = ([x_i - r, x_i + r] \times [0, y_i]) \setminus [R(x_{i-1}, y_{i-1}) \cup R(x_{i+1}, y_{i+1})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 2 \le i \le (n-1) \\ \mathcal{D}(x_n, y_n) = ([x_n - r, x_n] \times [0, y_n]) \setminus R(x_{n-1}, y_{n-1}). \end{cases}$$

In other words, $\cup_{i=1}^n \mathcal{D}(x_i, y_i)$ is the domain of the possible erasers of the points (x_i, y_i), $1 \le i \le n$, and the probability p decreases with the ν-measure of this domain.

Proof. (i) Using Theorem 14, it suffices to obtain that for any measurable bounded function h on $\mathbb{R}^2$,

$$\mathbf{E}\{h(\eta_1, \rho'_1)\mathbf{1}_{B'_1}\} = \mathbf{P}(B'_1) \int h(u, v)\varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}(u, v)dudv. \tag{70}$$

Taking the conditional expectation of $h(\eta_1, \rho'_1)\mathbf{1}_{B'_1}$ with respect to $\sigma(\eta_1, \rho'_1)$, we have

$$\begin{aligned} \mathbf{E}\{h(\eta_1, \rho'_1)\mathbf{1}_{B'_1}\} &= \mathbf{E}\{h(\eta_1, \rho'_1)e^{-(\eta_1 \wedge r)F(\rho'_1)}\} \\ &= C\int_0^{+\infty} du \int_0^\varepsilon h(u, v) \left(\mathbf{1}_{[0,r]}(u)e^{-uF(v)}\right. \\ &\qquad\qquad \left. + \mathbf{1}_{(r,+\infty)}(u)e^{-(u-r)F(\varepsilon)}e^{-rF(v)}\right) \alpha(F(v))f(v)dv \\ &= C\int\int h(u, v)\varphi_{(X_1^\varepsilon, Y_1^\varepsilon)}(u, v)dudv, \end{aligned}$$

where C is a positive constant. Consequently, we get equality (70).
(ii) Let g and h be two Borel bounded functions defined on $\mathbb{R}^{2n}$, respectively $\mathbb{R}^2$. Let us prove the following equality, for any $n \ge 1$:

$$\begin{aligned} &\mathbf{E}\{g(\eta_1, \rho_1, r + \eta_2, \rho_2, \cdots, r + \eta_n, \rho_n)h(r + \eta_{n+1}, \rho'_{n+1})\mathbf{1}_{\cap_{i=1}^n B_i}\mathbf{1}_{B'_n}\} \\ &= C'\mathbf{E}\left\{g(\eta_1, \rho_1, r + \eta_2, \rho_2, \cdots, r + \eta_n, \rho_n)\mathbf{1}_{\cap_{i=1}^n B_i}\alpha(F(\rho_n))\right. \\ &\qquad\qquad \left.\int h(u, v)\theta^\varepsilon(\rho_n; u, v)dudv\right\}, \end{aligned} \tag{71}$$

where C' is a positive constant.

We notice that using a reasoning by induction and Theorem 15, (71) implies the result (ii) of Theorem 18.

Since

$$\varphi_{\rho'}(x) = \frac{F(\varepsilon)}{\int_0^{F(\varepsilon)} \alpha(v)dv}\alpha(F(x))\varphi_\rho(x),$$

the right hand side of (71) is equal to

$$C'\mathbf{E}\left\{g(\eta_1, \rho_1, r + \eta_2, \rho_2, \cdots, r + \eta_n, \rho'_n)\mathbf{1}_{\cap_{i=1}^n B_i} \int h(u, v)\theta^\varepsilon(\rho'_n; u, v)dudv\right\}.$$

Let us prove (71).

We take the conditional expectation with respect to $\sigma(\eta_1, \rho_1, A_1, \cdots, \eta_n, \rho_n, A_n)$:

$$\begin{aligned}
&\mathbf{E}\{g(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_n, \rho_n) h(r+\eta_{n+1}, \rho'_{n+1}) \mathbf{1}_{\cap_{i=1}^n B_i} \mathbf{1}_{B'_n}\} \\
&\qquad = \mathbf{E}\left\{g(\eta_1, \rho_1, r+\eta_2, \rho_2, \cdots, r+\eta_n, \rho_n) \mathbf{1}_{\cap_{i=1}^n B_i} \mathbf{E}(h(r+\eta_{n+1}, \rho'_{n+1}) \mathbf{1}_{B'_n} | \rho_n)\right\}. \quad (72)
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{E}(h(r+\eta_{n+1}, \rho'_{n+1}) \mathbf{1}_{B'_n} | \rho_n) \\
&\quad = \int\int h(r+u, v) e^{-(u\wedge r)(F(v)\vee F(\rho_n)) - r(F(v)\wedge F(\rho_n))} \varphi_\eta(u) \varphi_{\rho'}(v) du dv \\
&\quad = K \int_r^{+\infty} du \int_0^\varepsilon h(u, v) \left[\mathbf{1}_{r\leq u\leq 2r} \left\{\mathbf{1}_{v>\rho_n} e^{-(u-r)F(v) - rF(\rho_n)} + \mathbf{1}_{v\leq\rho_n} e^{-(u-r)F(\rho_n) - rF(v)}\right\}\right. \\
&\qquad\qquad \left. + \mathbf{1}_{u>2r} e^{-(u-2r)F(\varepsilon)} e^{-r(F(\rho_n)+F(v))}\right] \alpha(F(v)) f(v) dv \\
&\quad = K \int\int h(u, v) \theta^\varepsilon(\rho_n; u, v) \alpha(F(\rho_n)) du dv \text{ a.s.}, \qquad (73)
\end{aligned}$$

where K is a positive constant. Inserting the equality (73) in (72), we get (71).

□

5 The saturation case

Let us define the process

$$\Lambda_\infty = \{x \in \mathbb{R}; \exists y \geq 0 | (x, y) \in \Psi\},$$

where Ψ is the point process on $\mathbb{R} \times \mathbb{R}_+$ defined in the first section. Λ_∞ is a saturation model in the sense that no new crack can be added.

It is immediate that Λ_ε converges in law [14] to Λ_∞ when $\varepsilon \to +\infty$. We prove that λ_ε tends to the mean crack number λ_∞ and D_ε (resp. $|I_0^\varepsilon|$, L_ε) converges in distribution to D_∞ (resp. $|I_0^\infty|$, L_∞).

Let us define in the same way the saturation process Λ_∞^+ of the cracks on the positive half-line, $X_1^\infty, \cdots, X_n^\infty, \cdots$ being the successive crack positions of this process and $Y_1^\infty, \cdots, Y_n^\infty, \cdots$ the corresponding strain levels at which they occur. We determine the joint distribution of

$$\{(X_i^\infty, Y_i^\infty); 1 \leq i \leq n\}, \quad n \geq 1.$$

We also provide a result of convergence in distribution of $(X_{n+1}^\infty - X_n^\infty, Y_n^\infty)$ when $n \to +\infty$.

Let us start with λ_∞ and D_∞ (resp. $|I_0^\infty|$, L_∞).

Theorem 20 *When ε goes to infinity (saturation), we have*

(i) $\lambda_\varepsilon \to \lambda_\infty = \int_0^{+\infty} \alpha(v)^2 dv$;

(ii) a) $(D_\varepsilon, L_\varepsilon)$ converges in distribution, as $\varepsilon \to +\infty$, to (D_∞, L_∞). The two-dimensional r.v. (D_∞, L_∞) has a density:

$$\varphi_{(D_\infty,L_\infty)}(y,u) = \varphi_{L_\infty}(y)\Pi^{D_\infty}(y;u), \tag{74}$$

where φ_{L_∞} (resp. $\Pi^{D_\infty}(y;\cdot)$) is the density of L_∞ (resp. the conditional density of D_∞ given $\{L_\infty = y\}$) and

$$\begin{aligned}\varphi_{L_\infty}(y) &= \frac{1}{\lambda_\infty}\alpha(F(y))^2 f(y)\mathbf{1}_{\mathbb{R}_+}(y), && (75)\\ \Pi^{D_\infty}(y;u) &= \mathbf{1}_{\{r\le u\le 2r\}}\left\{F(y)e^{-(u-r)F(y)} + \frac{e^{-rF(y)}}{\alpha(F(y))}\int_{F(y)}^{+\infty} e^{-(u-r)v}\alpha(v)dv\right\}; && (76)\end{aligned}$$

b) In particular the density φ_{D_∞} of D_∞ is:

$$\varphi_{D_\infty}(x) = \frac{2}{\lambda_\infty}\mathbf{1}_{[r,2r]}(x)\int_0^{+\infty} e^{-(x-r)v} v\alpha(v)^2 dv; \tag{77}$$

(iii) $|I_0^\varepsilon|$ converges in distribution to $|I_0^\infty|$ of density $\varphi_{|I_0^\infty|}$ such that

$$\varphi_{|I_0^\infty|}(x) = 2x\int_0^{+\infty} e^{-(x-r)v} v\alpha(v)^2 dv \cdot \mathbf{1}_{[r,2r]}(x). \tag{78}$$

Remark 21 (i) In a different theoretical context, Rényi gave an equivalent formulation of the point (i) in [17] (see result (0.10)). He obtained that the mean crack number at saturation is 0.748 approximately (for $r = 1$). To our knowledge, the other results of Theorem 20 are new.
(ii) Let us notice that the distribution of D_∞ has a decreasing density on $[r, 2r]$ and an easy calculation provides the following equality, similar to (13):

$$\mathbf{E}D_\infty = \frac{1}{\lambda_\infty}.$$

Proof. The point (i) is immediate.

As for point (ii), let us determine an equivalent of $\alpha(t)$, when $t \to +\infty$.

We decompose in $\alpha(t)$, the integral over $[0, t]$ as a sum of an integral over $[0, 1]$ plus an integral over $[1, t]$. We obtain:

$$\alpha(t) \;\sim\; \frac{\alpha_0}{t}, \text{ when } t \to +\infty, \tag{79}$$

where $\alpha_0 = (1/r)\exp\left\{-\int_0^1 \frac{1-e^{-s}}{s}ds + \int_1^{+\infty}\frac{e^{-s}}{s}ds\right\}$.

The point (ii) of Theorem 20 then follows easily from Theorem 13 and (79). Taking the limit in the result of Proposition 12, we deduce (iii).

□

By taking the limit when $\varepsilon \to +\infty$ in the formulas of Theorems 14 and 15, we then get the following results in the saturation case concerning the distribution of $\{(X_i^\infty, Y_i^\infty); 1 \leq i \leq n\}$, $n \geq 1$.

Theorem 22 *(i) (X_1^∞, Y_1^∞) is a two-dimensional r.v. with density:*

$$\varphi_{(X_1^\infty, Y_1^\infty)}(u,v) = e^{-uF(v)}\alpha(F(v))f(v)\mathbf{1}_{\{0\leq u\leq r\}}\mathbf{1}_{\{v\geq 0\}};$$

(ii) Conditionally to $(X_1^\infty, \cdots, X_n^\infty, Y_1^\infty, \cdots, Y_n^\infty)$, the couple $(X_{n+1}^\infty - X_n^\infty, Y_{n+1}^\infty)$ has a density $\theta^\infty(Y_n^\infty; \cdot)$ such that for every $y, u, v \geq 0$:

$$\begin{aligned}&\theta^\infty(y;u,v)\\ &\quad = \left(e^{-(u-r)F(y)}e^{-rF(v)}\mathbf{1}_{\{0\leq v\leq y\}} + e^{-rF(y)}e^{-(u-r)F(v)}\mathbf{1}_{\{v>y\}}\right)\frac{\alpha(F(v))}{\alpha(F(y))}f(v)\mathbf{1}_{\{r\leq u\leq 2r\}}.\end{aligned}$$

Remark 23 (i) Since the process Λ_∞^+ corresponds to saturation, we obviously have

$$X_1^\infty \leq r \text{ and } (X_{n+1}^\infty - X_n^\infty) \in [r, 2r], \quad \forall n \geq 1 \ a.s..$$

However we are not able to prove, as we did in Theorem 18, that $(X_n^\infty)_{n\geq 1}$ is a conditioned renewal process.
(ii) As in the non-saturated case, it suffices to have the law of (X_1^∞, Y_1^∞) on the one hand and the law of $(Y_n^\infty)_{n\geq 1}$ on the other to determine the positions $(X_n^\infty)_{n\geq 1}$.

Theorem 16 can be easily generalized in the following way.

Theorem 24 *$(Y_n^\infty)_{n\geq 1}$ is a homogeneous Markov chain such that:*

(i) Y_1^∞ has a density $\varphi_{Y_1^\infty}$ such that

$$\varphi_{Y_1^\infty}(y) = \frac{1-e^{-rF(y)}}{F(y)}\alpha(F(y))f(y)\mathbf{1}_{\mathbb{R}_+}(y);$$

(ii) the law of Y_{n+1}^∞ conditionally to $\{Y_n^\infty = y\}$, $y \geq 0$, is independent from n and has a density $\Pi^{Y,\infty}(y;\cdot)$ such that for every $v \geq 0$,

$$\begin{aligned}&\Pi^{Y,\infty}(y;v)\\ &\quad = \left(\mathbf{1}_{\{0\leq v\leq y\}}(1-e^{-rF(y)})\frac{e^{-rF(v)}}{F(y)} + \mathbf{1}_{\{y<v\}}e^{-rF(y)}\frac{1-e^{-rF(v)}}{F(v)}\right)\frac{\alpha(F(v))}{\alpha(F(y))}f(v);\end{aligned}$$

(iii) the stationary law of $(Y_n^\infty)_{n\geq 1}$ is the distribution of L_∞, with density given by (75);

(iv) conditionally to $(X_1^\infty, Y_1^\infty, \cdots, X_n^\infty, Y_n^\infty)$, the distribution of the r.v. $(X_{n+1}^\infty - X_n^\infty)$ has a density which depends only on Y_n^∞ and is equal to $\Pi^{D_\infty}(Y_n^\infty;\cdot)$ (recall that $\Pi^{D_\infty}(y;\cdot)$ is given by (76)).

We finally generalize Theorem 17.

Theorem 25 *The couple $(X_{n+1}^\infty - X_n^\infty, Y_n^\infty)$ converges in law when $n \to +\infty$, and the limit distribution is the law of (D_∞, L_∞) (cf Theorem 20).*

References

[1] M. F. Ashby and D. R. Jones. *International Series on Materials Science and Technology*, volume 39. Pergamon, New York, 1986.

[2] J. Berréhar, C. Lapersonne-Meyer, M. Schott, and J. Villain. Formation of periodic crack structures in polydiacetylene single crystal thin films. *J. de Physique France*, 50:923–935, 1989.

[3] S. L. Bucklow. The description and classification of craquelure. *Studies in Conservation*, 44:233–244, 1999.

[4] J. P. Chilès. Fractal and geostatistical methods for modeling of a fracture network. *Mathematical Geology*, 20(6):631–654, 1988.

[5] R. Durrett. *Probability: theory and examples.* Duxbury Press, Belmont, CA, second edition, 1996.

[6] G. Gille. Investigations on mechanical behaviour of brittle wear-resistant coatings: II Theory. *Thin Solid Films*, 111:201–218, 1984.

[7] P. A. Gillespie, C. B. Howard, J. J. Walsh, and J. Watterson. Measurement and characterization of spatial distributions of fractures. *Tectonophysics*, 226:113–141, 1993.

[8] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[9] Y. Leterrier, D. Pellaton, D. Mendels, R. Glauser, J. Andersons, and J. A. Manson. Biaxial fragmentation on thin silicon oxide coatings on poly(ethylene terephtalate). *J. Mat. Sci.*, 36:2213–2225, 2001.

[10] D. Mannion. Random packing of an interval. *Advances in Appl. Probability*, 8(3):477–501, 1976.

[11] A. Mézin, J. Lepage, N. Pacia, and D. Paulmier. Étude statistique de la fissuration des revêtements I: Théorie. *Thin Solid Films*, 172:197–209, 1989.

[12] A. Mézin and P. Vallois. Statistical analysis of unidirectional multicracking of coatings by a two-dimensional Poisson point process. *Math. Mech. Solids*, 5(4):417–440, 2000.

[13] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[14] J. Neveu. Processus ponctuels. In *École d'Été de Probabilités de Saint-Flour, VI—1976*, pages 249–445. Lecture Notes in Math., Vol. 598. Springer-Verlag, Berlin, 1977.

[15] P. E. Ney. A random interval filling problem. *Ann. Math. Statist.*, 33:702–718, 1962.

[16] M. D. Penrose. Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.*, 218(1):153–176, 2001.

[17] A. Rényi. On a one-dimensional problem concerning random space filling. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 3(no 1/2):109–127, 1958.

[18] R. Y. Rubinstein. *Simulation and the Monte Carlo method*. John Wiley & Sons Inc., New York, 1981. Wiley Series in Probability and Mathematical Statistics.

[19] B. Widom. Random sequential addition of hard spheres to a volume. *J. Chem. Phys.*, 44(10):3888–3894, 1966.

[20] N. Wiener. The ergodic theorem. *Duke Math.*, 5:1–18, 1939.

Chapitre 7

Annexes.

7.1 An elementary proof of the equality $\mathbf{E}N_0(\mathcal{C}) = 6$.

Let Φ be a homogeneous Poisson point process of intensity measure the Lebesggue measure on $\mathbb{R}^2$. Let us recall that for every $n \in \mathbb{N}$,

$$\Phi \cap D(0,n) \stackrel{law}{=} \{X_1, \cdots, X_{N_n}\},$$

where $\{X_i; i \geq 1\}$ is a sequence of i.i.d random variables which are uniformly distributed on the disk $D(0,n)$, and N_n is an independent Poisson variable of mean value πn^2.

Let Γ be the Voronoi diagram constructed with the set of nuclei $\mathcal{N} = \{X_1, \cdots, X_{N_n}\}$. We will denote by $C(x)$ the cell associated to $x \in \mathcal{N}$.

Lemma 7.1.1 *$C(x)$ is not bounded if and only if x is on the boundary of the convex hull of $\mathcal{N}$.*

Proof. Suppose $C(x)$ is not bounded and let $D_+ = (x + \mathbb{R}_+ u_\theta)$ be a half-line included in $C(x)$ (where u_θ is a unit vector of direction $\theta \in [0, 2\pi)$). If x is not on the boundary of the convex hull of $\mathcal{N}$, let $[y,z]$ be the segment of the boundary that D_+ cuts. As the angle $(\overrightarrow{xy}, \overrightarrow{xz})$ is less that π, the bisecting line of either $[x,y]$ or $[x,z]$ cuts D_+. So D_+ is not completely contained in $C(x)$, which is absurd.

Conversely, if x is on the boundary of the convex hull of $\mathcal{N}$, let D_1 and D_2 be the two half-lines starting from x which are outer normal to the two sides of the boundary containing x. The cone between D_1 and D_2 then is contained in $C(x)$ because x is the projection on the convex hull of every point of the cone. Consequently, $C(x)$ is not bounded.

$\square$

We denote by B_n the number of unbounded cells, V_n the number of vertices of the graph Γ and E_n the number of edges. We have two relations connecting these numbers : le us first recall that the Voronoi diagram is *normal*, in the sense that a particular vertex (respectively edge) belongs to exactly three (resp. two) cells. Consequently, we have

$$2E_n = 3V_n + B_n. \tag{7.1}$$

Secondly, let us add a point at infinity and connect each unbounded side to this point. We then get a connected planar graph $\widehat{\Gamma}$ with (V_n+1) vertices, E_n edges and N_n connex domains. Applying Euler's formula, we deduce that

$$V_n + 1 - E_n + N_n = 2. \tag{7.2}$$

Inserting (7.1) in (7.2), we deduce that when $N_n \geq 3$,

$$\frac{3V_n}{N_n} \;=\; 6 - 3\frac{B_n}{N_n} - \frac{6}{N_n} \quad \text{a.s..} \tag{7.3}$$

Let us focus on the two last terms of the right-hand side of (7.3). Since N_n is a Poisson variable of mean value πn^2,

$$\frac{1}{N_n} \xrightarrow{\mathbf{P}} 0 \text{ when } n \to +\infty, \tag{7.4}$$

where $\xrightarrow{\mathbf{P}}$ denotes the convergence in probability. In addition, conditionally to $\{N_n = k\}$, $k \geq 3$, B_n is equal in law to the number C_k of points on the boundary of the convex hull of k points uniformly distributed on a disk. We know from [47] that there exists a constant $a > 0$ such that

$$\frac{C_k}{ak^{1/3}} \xrightarrow{\mathbf{P}} 0 \text{ when } n \to +\infty. \tag{7.5}$$

In particular, let us fix $\varepsilon > 0$. We deduce from (7.5) that

$$\mathbf{P}\left\{\frac{C_k}{k} \geq \varepsilon\right\} \to 0 \text{ when } k \to +\infty,$$

so let $K \geq 3$ be such that $\mathbf{P}\left\{\frac{C_K}{K} \geq \varepsilon\right\} \leq \varepsilon/2$. We then obtain that

$$\begin{aligned}
\mathbf{P}\left\{\frac{B_n}{N_n} \geq \varepsilon | N_n \geq 3\right\} &= \sum_{k\geq 3} \mathbf{P}\left\{\frac{C_k}{k} \geq \varepsilon\right\} e^{-\pi n^2}\frac{(\pi n^2)^k}{k!} \\
&\leq \mathbf{P}\left\{\frac{C_K}{K} \geq \varepsilon\right\} + e^{-\pi n^2}\sum_{k=0}^{K}\frac{(\pi n^2)^k}{k!} \\
&\leq \frac{\varepsilon}{2} + e^{-\pi n^2}\sum_{k=0}^{K}\frac{(\pi n^2)^k}{k!} \leq \varepsilon \text{ when n is large enough.}
\end{aligned}$$

Consequently, we have proved that

$$\frac{B_n}{N_n} \xrightarrow{\mathbf{P}} 0 \text{ when } n \to +\infty. \tag{7.6}$$

Inserting the results of convergence (7.4) and (7.6) in (7.3), we get that $3V_n/N_n$, which is more or less the number of vertices per cell in the disk $D(0,n)$ converges in probability, i.e.

$$\frac{3V_n}{N_n} \xrightarrow{\mathbf{P}} 6 \text{ when } n \to +\infty.$$

On the spectral function of the Johnson-Mehl cells. *

André Goldman and Pierre Calka[†]

Abstract

Denote by $\varphi(t) = \sum_{n\geq 1} e^{-\lambda_n t}$, $t > 0$, the spectral function related to the Dirichlet laplacian for the typical cell $\mathcal{C}$ of a Johnson-Mehl tessellation in $\mathbb{R}^d$, $d \geq 2$. As in the particular Voronoi case (see [5]), we show that the expectation $\mathbf{E}\varphi(t)$, $t > 0$, is a functional of the convex hull of a standard d-dimensional Brownian bridge. This enables us to study the asymptotic behaviour of $\mathbf{E}\varphi(t)$, when $t \to 0^+$ under suitable integrability conditions.

Introduction.

The Johnson-Mehl tessellation was introduced in 1939 [6] as a spatial model for crystals in metallic systems generated by growing random nuclei. Thus, consider a space-time point process

$$\Phi = \{a_i = (x_i, t_i) \in \mathbb{R}^d \times \mathbb{R}_+, i \geq 1\}, \quad d \geq 2.$$

A nucleus $x_i \in \mathbb{R}^d$ is born at time $t_i \geq 0$ and then starts to grow with a constant speed $v > 0$, so that $y \in \mathbb{R}^d$ is reached at time

$$T_i(y) = t_i + ||x_i - y||/v. \tag{1}$$

Let

$$C_i = \{y \in \mathbb{R}^d; T_i(y) \leq T_j(y) \quad \forall j \neq i\}. \tag{2}$$

be the associated cell. The set of non-empty cells C_i constitutes the Johnson-Mehl tessellation [2]. Throughout this work, Φ is assumed to be a Poisson point process with intensity measure $dx\Lambda(dt)$, where dx denotes the Lebesgue measure on $\mathbb{R}^d$ and Λ is a locally finite measure satisfying the "canonical" conditions of J. Møller [9]:

$$\Lambda([0,\infty)) > 0 \tag{3}$$

American Mathematical Society 2000 subject classifications. Primary 35P15, 60D05, 60J65; secondary 52A22, 60G55.

Key words and phrases. Brownian bridge, convex hull, eigenvalues, empirical distributions, Johnson-Mehl tessellations, Laplace operator, Palm distribution, spectral function, stochastic geometry, Voronoi tessellations.

†*Postal address*: Université Claude Bernard Lyon 1, LaPCS, Bât. B, Domaine de Gerland, 50, avenue Tony-Garnier F-69366 Lyon Cedex 07, France. *E-mail*: Andre.Goldman@univ-lyon1.fr, Pierre.Calka@univ-lyon1.fr

$$\lambda = \int p(t)\Lambda(dt) < \infty \tag{4}$$

where

$$p(t) = \exp\left(-v^d\omega_d \int_0^t (t-w)^d \Lambda(dw)\right)$$

and $\omega_d = \pi^{d/2}/\Gamma(\frac{1}{2}d+1)$ is the volume of the unit ball $B(1)$ in $\mathbb{R}^d$. Without any loss of generality we will choose $v = 1$.

The condition (4) means that the number of non-empty cells, whose associated nuclei belong to an arbitrarily fixed Borel subset of $\mathbb{R}^d$ of finite d-dimensional Lebesgue measure, has finite expectation.

Remark that in the particular case when Λ is a Dirac measure, that means in the case when all nuclei are born at the same time, we obtain the classical Poisson-Voronoi tessellation [8],[10],[12]. Let us note also that the case $d = 1$, that trivializes the geometric structure of cells (which become intervals) is of some interest too; for example as a stochastic model for DNA replication [1].

A systematic and rigorous study of fundamental properties of the Johnson-Mehl tessellation has been done by J. Møller [9]. Though, the explicit distributions of principal geometrical characteristics of the tessellation are still unknown and therefore simulation procedures have been developed [11].

Throughout this work, we investigate the expectation of the spectral function of the Johnson-Mehl typical cell by using a method relying on the Brownian motion and developed in [5] for the Poisson-Voronoi tessellation.

Denote by

$$\varphi(t) = \sum_{n\geq 1} \exp(-t\lambda_n), t > 0, \tag{5}$$

the spectral function of the Johnson-Mehl typical cell $\mathcal{C} \subset \mathbb{R}^d$ where

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots,$$

are the (random) eigenvalues of the Laplacian under Dirichlet boundary conditions. Clearly, the spectral function $\varphi(t)$, $t > 0$, coincides with the Laplace transform of the distribution function of eigenvalues:

$$N(t) = \sum_{n\geq 1} 1_{\{\lambda_n \leq t\}}, t > 0. \tag{6}$$

Besides, let

$$(W_t(s))_{0\leq s\leq t}, \quad W_t(0) \equiv W_t(t) \equiv 0, t > 0, \tag{7}$$

be a d-dimensional Brownian bridge on the interval $0 \leq s \leq t$, starting at the origin and independent of the Johnson-Mehl tessellation. Consider then

$$\mathbf{W}(t) = \{W_t(s); 0 \leq s \leq t\} \subset \mathbb{R}^d \tag{8}$$

the associated path and $\widehat{\mathbf{W}}(t)$ its closed convex hull. If $t = 1$, we note $\mathbf{W}(1) = \mathbf{W}$.

Moreover, for all Borel set $C \subset \mathbb{R}^d$, let us denote by $V_d(C, x, s)$, $x \in \mathbb{R}^d$, $s \in \mathbb{R}$, the d-dimensional Lebesgue measure of the set

$$\bigcup_{\substack{y \in C \\ ||y-x||+s>0}} B(y, ||y-x|| + s),$$

where $B(x, r)$ is the euclidean ball centered at $x \in \mathbb{R}^d$, and of radius $r > 0$.

We note also $V_d(C, x) = V_d(C, x, 0)$.

We prove that the expectation of the spectral function has the following form:

$$\mathbf{E}\varphi(t) = \frac{1}{\lambda(4\pi t)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \exp\left(- \int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right) \right\} dx \Lambda(du), \quad (9)$$

where $\overline{\mathbf{E}}$ is the expectation associated to the Brownian bridge.

In the Voronoi case [5], we obtained a theoretical expansion of $\mathbf{E}\varphi(t)$ when $t \to 0^+$ at any order which is explicit for the first three terms. In the general case of Johnson-Mehl tessellations, the study of volume $V_d(\sqrt{2t}\mathbf{W}(1), x, s)$ is a more delicate matter so we limited our investigation to the second order. For measures Λ satisfying some mild regularity conditions including usual cases, we prove that:

$$\mathbf{E}\varphi(t) = \frac{1}{\lambda(4\pi t)^{\frac{d}{2}}} - \frac{\pi^{d/2}}{\lambda\Gamma(d-\frac{1}{2})t^{\frac{d-1}{2}}} \int_0^\infty \left\{ \int_0^t (t-s)^{d-1}\Lambda(ds) \right\}^2 p(t)dt + O(\frac{1}{t^{\frac{d}{2}-1}}). \quad (10)$$

It is well-known (see [7], [13], [16]) that for a bounded domain $D \subset \mathbb{R}^d$ the associated spectral function has asymptotic expansion of the form

$$\varphi_D(t) = \frac{V_d(D)}{(4\pi t)^{\frac{d}{2}}} - \frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})}{4\pi^{\frac{d}{2}}(d-1)!t^{\frac{d-1}{2}}} V_{d-1}(D) + O(\frac{1}{t^{\frac{d}{2}-1}}), \quad \text{as } t \longrightarrow 0^+, \quad (11)$$

where $V_{d-1}(D)$ denotes the $(d-1)$-dimensional Hausdorff measure of the boundary of D.

Consequently we may anticipate the geometric significance of the two coefficients of the expansion (10). Indeed, we recover from Møller's work [9] that for the Johnson-Mehl typical cell

$$\mathbf{E}(V_d(\mathcal{C})) = \frac{1}{\lambda}, \quad \mathbf{E}(V_{d-1}(\mathcal{C})) = \frac{2}{\lambda} \frac{2\pi^d (d-1)!}{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})\Gamma(d-\frac{1}{2})} \int_0^\infty \left\{ \int_0^t (t-s)^{d-1}\Lambda(ds) \right\}^2 p(t)dt. \quad (12)$$

1 Preliminaries.

1.1 The Johnson-Mehl tessellation.

The conditions of J. Møller (3) et (4) imply (see [9] Prop.3.1.) that the following consistency conditions are verified:

(P1) For any $(c,t) \in \Phi$ and any vector $u \in \mathbb{R}^d$ there exists $(y,s) \in \Phi$ with

$$(y-x)\cdot u > v(s-t),$$

where $\cdot$ is the usual inner product in $\mathbb{R}^d$.

(P2) If $i,j \geq 1, i \neq j$, then $t_j \neq T_i(x_j)$.

(P3) $(d+1)$ distinct nuclei are always affinely independent.

(P4) A fixed point $x \in \mathbb{R}^d$ can be reached at the same time by at most $(d+1)$ growing nuclei.

The non-empty cells C_i, defined by (2), are then almost surely closed, bounded and star-shaped sets; all of them are almost surely convex if and only if Λ is a Dirac measure. The set of these cells constitutes a tessellation of $\mathbb{R}^d$, that means:

- the cells are space-filling: $\bigcup_i C_i = \mathbb{R}^d$;
- they have disjoint interiors;
- they are regular: $C_i = \mathrm{cl}(\mathrm{int}\ C_i)$.

Moreover, for $1 \leq k \leq d$, any non-empty k-dimensional face (or k-face) defined as the intersection of $(d-k+1)$ cells, has Hausdorff dimension k (see [9] Prop. 3.1.). This implies in particular the normality of the tessellation, that means any non-empty k-dimensional face is contained in the boundary of exactly $\binom{d-k+1}{l-k}$ non-empty l-dimensional faces, $0 \leq k \leq l \leq d$.

1.2 The Johnson-Mehl typical cell.

Let $\mathcal{K}$ be the set of compact sets of $\mathbb{R}^d, d \geq 1$, endowed with the usual Hausdorff metric. The Johnson-Mehl typical cell $\mathcal{C}$ is a $\mathcal{K}$-valued random variable whose distribution (the Palm measure), is well defined by the formula (see [9]):

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\lambda V_d(B)}\mathbf{E}\sum_{i:x_i\in B, C_i\neq\emptyset} h(C_i - x_i), \tag{13}$$

for all measurable function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$, and Borel sets $B \subset \mathbb{R}^d$, $0 < V_d(B) < \infty$, where $V_d(B)$ denotes the d-dimensional Lebesgue measure of B.

More generally, we have Campbell's formula:

$$\int \mathbf{E}f(\mathcal{C}, y)dy = \frac{1}{\lambda}\mathbf{E}\sum_{i:C_i\neq\emptyset} f(C_i - x_i, x_i), \tag{14}$$

for all measurable function $f : \mathcal{K} \times \mathbb{R}^d \longrightarrow \mathbb{R}_+$.

Consider a random "birth time" $\tau \geq 0$ independent of the process Φ and whose distribution is $(p(t)/\lambda)\Lambda(dt)$. The typical cell can be interpreted (see [9], 3.7.) as the cell

$C\left((0,\tau)|\Phi_\tau \cup \{(0,\tau)\}\right)$ associated with a nucleus starting to grow at the origin at time $\tau \geq 0$ when we add the point $(0,\tau)$ to the conditional process

$$\Phi_\tau = \{\Phi| \text{ the origin is not covered by any growing nucleus at time } \tau\}.$$

Remark that this result includes the well-known fact that the typical cell of the Poisson-Voronoi tessellation (corresponding to the choice $\Lambda = \delta_0$) coincides in law with the cell associated with the nucleus based at the origin when we add the origin to the point process Φ.

For all Borel set $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, we define $V_d(A,x,t) = V_d(A-x,0,t)$ as the d-dimensional area of the set

$$\bigcup_{\substack{y\in A\\ ||y-x||+t>0}} B(y, ||y-x||+t).$$

We have then:

Proposition 1 *Consider a bounded Borel set $A \subset \mathbb{R}^d$. Then*

$$\mathbf{P}\{A \subset \mathcal{C}\} = \frac{1}{\lambda}\int_0^\infty \exp\left\{-\int_0^\infty V_d(A,0,u-w)\Lambda(dw)\right\}\Lambda(du).$$

Proof. Let us recall first that for all fixed $u > 0$, the conditioned process

$$\Phi_u = \{\Phi| \text{ the origin is not covered at time u}\},$$

is a Poisson point process of intensity measure

$$1_{\{(z,w)\in\mathbb{R}^d\times\mathbb{R}_+;||z||+w>u\}}dz\Lambda(dw).$$

Using the equality in law

$$\mathcal{C} \stackrel{law}{=} C\left((0,\tau)|\Phi_\tau \cup \{(0,\tau)\}\right),$$

we have that

$$\begin{aligned}
&\mathbf{P}\{A \subset \mathcal{C}\}\\
&= \int_0^\infty \mathbf{P}\left\{A \subset C\left((0,u)|\Phi_u \cup \{(0,u)\}\right)\right\}\frac{p(u)}{\lambda}\Lambda(du)\\
&= \int_0^\infty \exp\left\{-\int_{\mathbb{R}^d}\int_0^\infty 1_{\{||y-z||+w<||y||+u,\forall y\in A\}}1_{\{||z||+w>u\}}\Lambda(dw)dz\right\}\frac{p(u)}{\lambda}\Lambda(du)\\
&= \int_0^\infty \exp\left\{-\int_0^\infty V_d(A,0,u-w)\Lambda(dw) + \int_0^u \omega_d(u-w)^d\Lambda(dw)\right\}\frac{p(u)}{\lambda}\Lambda(du)\\
&= \frac{1}{\lambda}\int_0^\infty \exp\left\{-\int_0^\infty V_d(A,0,u-w)\Lambda(dw)\right\}\Lambda(du).
\end{aligned}$$

$\square$

The values of the expectations of the principal geometrical characteristics of the typical cell are known [9]. We call $V_{d-1}(\mathcal{C})$ the $(d-1)$-dimensional area of the boundary $\partial\mathcal{C} = \mathrm{cl}(\mathcal{C}) \setminus \mathrm{int}(\mathcal{C})$ and $N_0(\mathcal{C})$ the number of 0-dimensional faces of $\mathcal{C}$ (defined in [9]). We have (see [9] and [10]):

Proposition 2 **(i)** $\mathbf{E}V_d(\mathcal{C}) = \frac{1}{\lambda}$,

(ii) $\mathbf{E}V_{d-1}(\mathcal{C}) = \frac{2}{\lambda}\frac{2\pi^d(d-1)!}{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})\Gamma(d-\frac{1}{2})}\int_0^\infty \left\{\int_0^t (t-s)^{d-1}\Lambda(ds)\right\}^2 p(t)dt$,

(iii) $\mathbf{E}N_0(\mathcal{C}) = \frac{d+1}{\lambda}\frac{2^{d+1}\pi^{d(d+1)/2}\Gamma((d^2+1)/2)}{(d+1)!\Gamma(\frac{d^2}{2})\Gamma(\frac{d+1}{2})^d\Gamma(\frac{1}{2})}\int_0^\infty \left\{\int_0^t (t-s)^{d-1}\Lambda(ds)\right\}^{d+1} p(t)dt.$

We can notice that the condition (4) implies that $\mathbf{E}V_d(\mathcal{C}) < +\infty$ and $\mathbf{E}(V_{d-1}(\mathcal{C})) < \infty$, that means the $(d-1)$-dimensional area of the boundary $\partial\mathcal{C}$ of the typical cell is, in mean, finite. Besides, the condition

$$(\mathbf{C0}) \int_0^\infty \left(\int_0^t (t-s)^{d-1}\Lambda(ds)\right)^{d+1} p(t)dt < \infty$$

implies that the number of vertices of the typical cell $\mathcal{C}$ is finite in mean.

In the case when Λ is a Dirac measure (the Poisson-Voronoi case) or a finite linear combination of Dirac measures, the condition **(C0)** is satisfied. It is also satisfied with the choice $\Lambda(dt) = \alpha t^{\gamma-1}dt$, $\alpha \geq 0$, $\gamma \geq 0$ considered by J. Møller in [9] (the case $\alpha = 1$, $\gamma = 0$ being treated by E.N. Gilbert [2]) with the following numerical values:

$$\lambda = \frac{\alpha\Gamma(\gamma/(d+\gamma))}{(d+\gamma)[\alpha\omega_d\beta(d+1,\gamma)]^{\frac{\gamma}{d+\gamma}}}$$

$$\int_0^\infty \left\{\int_0^t (t-s)^{d-1}\Lambda(ds)\right\}^{m+1} p(t)dt$$
$$= (\alpha\beta(d,\gamma))^{m+1}\frac{\Gamma\left(\frac{(d+\gamma)(m+1)-m}{d+\gamma}\right)}{(d+\gamma)[\alpha\omega_d\beta(d+1,\gamma)]^{\frac{(d+\gamma)(m+1)-m}{d+\gamma}}}, \quad 1 \leq m \leq d,$$

where $\beta(.,.)$ denotes Euler's beta function .

On the other hand we can connect the typical cell with the cell C_0 containing the origin. More precisely, for all $x \in \mathbb{R}^d$ and for all $\omega \in \Omega$, let $C(x,\omega)$ be the cell containing the point x. We will note also $C(0,\omega) = C_0(\omega)$. For all fixed ω, $C(x,\omega)$ is well defined if x is not contained in the union of the boundaries of the cells $\bigcup_i \partial C_i$, which, according to what has been said, has d-dimensional Lebesgue measure equal to zero. Moreover, if $x \in \mathbb{R}^d$ is fixed, then we can see easily, by using the point (i) of Proposition 2 and J. Møller's argument in [10], p.67, that $C(x,\omega)$ is well defined almost surely. Thus, C_0 is almost surely well-defined and the result proved by Møller ([10], Prop. 3.3.2.) in the case of Voronoi tessellations can be generalised to any Johnson-Mehl tessellation, namely:

Lemma 1 *We have:*

$$\mathbf{E}h(\mathcal{C}) = \frac{1}{\mathbf{E}(1/V_d(C_0))}\mathbf{E}\left(\frac{h(C_0)}{V_d(C_0)}\right), \tag{15}$$

for all measurable and translation-invariant function $h : \mathcal{K} \longrightarrow \mathbb{R}_+$.

In particular,

$$\mathbf{E}\left(\frac{1}{V_d(C_0)}\right) = \lambda. \tag{16}$$

We now give the analogue to Proposition 1 for C_0.

Proposition 3 *Consider a bounded Borel set $A \subset \mathbb{R}^d$, containing the origin. Then*

$$\mathbf{P}\{A \subset C_0\} = \int_0^\infty \int \exp\left(-\int V_d(A, x, u-w)\Lambda(dw)\right) dx\Lambda(du).$$

Proof. By Campbell's formula (14) we obtain

$$\begin{aligned}\int \mathbf{P}\{-x + A \subset \mathcal{C}\}dx &= \frac{1}{\lambda}\mathbf{E}\sum_{i;C_i\neq\emptyset} 1_{\{-x_i+A\subset C_i-x_i\}} \\ &= \frac{1}{\lambda}\mathbf{P}\{A \subset C_0\}.\end{aligned}$$

It suffices now to use Proposition 1 to get the result.

□

From Proposition 3 we derive the following consequence:

Proposition 4 *We have:*

$$\mathbf{E}V_d(C_0) < \infty.$$

Proof. Indeed, by applying the preceding theorem to the Borel set $\{0\} \cup \{y\}$, $y \in \mathbb{R}^d$, we obtain:

$$\begin{aligned}\mathbf{E}V_d(C_0) &= \int \mathbf{P}\{y \in C_0\}dy \\ &= \int_0^\infty \int \exp\left(-\int_0^\infty V_d(\{0\} \cup \{y\}, x, u-w)\Lambda(dw)\right) dydx\Lambda(du). \end{aligned} \tag{17}$$

Consider now $T_0 > 0$ such that $\Lambda([0, T_0]) > 0$. Then, on one hand, we have:

$$\begin{aligned}&\int_{T_0}^\infty \int \exp\left(-\int_0^\infty V_d(\{0\} \cup \{y\}, x, u-w)\Lambda(dw)\right) dydx\Lambda(du) \\ &\leq \int_{T_0}^\infty \int \exp\left(-\omega_d \int_0^u [(||y-x|| + u - w)^d \vee (||x|| + u - w)^d]\Lambda(dw)\right) dydx\Lambda(du) \\ &= \int_{T_0}^\infty \int \exp\left(-\omega_d \int_0^u [(||y|| + u - w)^d \vee (||x|| + u - w)^d]\Lambda(dw)\right) dydx\Lambda(du) \\ &\leq 2\int_{T_0}^\infty \int_{\{||y||\leq||x||\}} \exp\left(-\omega_d \int_0^u (||x|| + u - w)^d\Lambda(dw)\right) dydx\Lambda(du) \\ &\leq 2\int_{T_0}^\infty \int_{\{||y||\leq||x||\}} \exp\left(-\omega_d||x||^d\Lambda([0,u]) - \omega_d \int_0^u (u-w)^d\Lambda(dw)\right) dydx\Lambda(du) \\ &\leq C\frac{1}{(\Lambda([0,T_0]))^2}\lambda < \infty, \end{aligned} \tag{18}$$

where $0 < C < +\infty$.

On the other hand,

$$\int_0^{T_0} \int \exp\left(- \int_0^{\infty} V_d(\{0\} \cup \{y\}, x, u-w)\Lambda(dw) \right) dydx\Lambda(du)$$
$$= 2 \int_0^{T_0} \int_{\{||y|| \le ||x||\}} \exp\left(- \int_0^{\infty} V_d(\{0\} \cup \{y+x\}, x, u-w)\Lambda(dw) \right) dydx\Lambda(du)$$
$$\le 2 \int_0^{T_0} \int_{\{||y|| \le ||x||\}} \exp\left(-\omega_d \int_0^{||x||+u} (||x|| + u - w)^d \Lambda(dw) \right) dydx\Lambda(du)$$
$$\le 2\Lambda([0,T_0]) \left\{ \int_{\{||y|| \le ||x|| \le 2T_0\}} dydx \right.$$
$$\left. + \omega_d \int_{\{||x|| > 2T_0\}} ||x||^d \exp\left(-\omega_d \int_0^{||x||} (||x|| - w)^d \Lambda(dw) \right) dx \right\}$$
$$\le C' + C'' \int_{2T_0}^{\infty} r^{2d-1} e^{-\omega_d \left(\frac{r}{2}\right)^d \Lambda([0,T_0])} dr < \infty. \tag{19}$$

where $0 < C' < +\infty$ and $0 < C'' < +\infty$ are constants.

The points (17), (18) and (19) imply the required result.

□

1.3 Some geometric estimations for the d-dimensional Brownian bridge.

Let us denote by $V_d(C,x) = V_d(C,x,0)$ for any bounded Borel set $C \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, $d \ge 2$. We obtained a precise formula for $V_d(C,x)$ when C is a convex set [4]. Moreover, we proved that there exists $t_0 > 0$ and a positive constant $L_d > 0$ such that for every $t \in (0,t_0)$, and $x \in \mathbb{R}^d$,

$$\overline{\mathbf{E}}\left(V_d(\widehat{\mathbf{W}}(2t), x) - \omega_d ||x||^d \right) - 2^{d-1}\pi^{d/2} \frac{\Gamma(d/2)}{\Gamma(d-1/2)} \sqrt{t} ||x||^{d-1}$$
$$-2^{d-1}(d-1)\pi^{(d-1)/2} \frac{\Gamma((d-1)/2)}{\Gamma(d-1)} t ||x||^{d-2} \le L_d t\sqrt{t}(1 + ||x||^{d-3}) \tag{20}$$

In addition, there exists an explicit constant $k_d > 0$ such that

$$\overline{\mathbf{E}}\left\{ (V_d(\widehat{\mathbf{W}}(2t), x) - \omega_d ||x||^d)^2 \right\} - 4^d ||x||^{2d-2} k_d 2t \le L_d t\sqrt{t}(1 + ||x||^{2d-3}). \tag{21}$$

In order to study the asymptotic behaviour of $V_d(\mathbf{W}(2t), x, s)$, when $T \to 0^+$, the principal idea is to compare the d-dimensional areas $V_d(\mathbf{W}(2t), x, s)$ and $V_d(\widehat{\mathbf{W}}(2t), f^s(x), 0)$, where $f^s(x) = x + s\frac{x}{||x||}$. We obtain:

Lemma 2 *There exists $t_0 > 0$ and a positive constant $K_d > 0$ such that:*

$$\overline{\mathbf{E}}\{V_d(\mathbf{W}(2t), x, s) - V_d(\widehat{\mathbf{W}}(2t), f^s(x))\} \leq K_d t \left(||x||^{d-3} + \frac{1}{||x||} \right) (1 + |s|^{d-1}) \tag{22}$$

for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$, satisfying $||x|| + s \geq 0$.

Lemma 3 *There exists $t_0 > 0$ and a positive constant $K'_d > 0$ such that:*

$$\overline{\mathbf{E}}\left\{ \left[V_d(\mathbf{W}(2t), x, s) - V_d(\widehat{\mathbf{W}}(2t), f^s(x)) \right]^2 \right\} \leq K'_d t(||x||^{2d-4} + 1)(1 + |s|^{2d-2})$$

for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$, such that $||x|| + s \geq 0$.

The proofs of these results are somewhat lengthy and technical. So, we postponed them to the Appendix.

Finally, as a consequence of Lemma 2 and (20), we obtain

$$\overline{\mathbf{E}}\left\{ V_d(\mathbf{W}(2t), x, s) - \omega_d(||x|| + s)^d \right\} \sim_{t \longrightarrow 0} \overline{\mathbf{E}} M_0 I_{d,1} \sqrt{2t} ||f^s(x)||^{d-1}, \tag{23}$$

which will be useful in the next section.

2 The spectral function of the Johnson-Mehl typical cell $\mathcal{C}$.

Let λ_n, $n \geq 1$, be the (random) eigenvalues of the Dirichlet laplacian for the Johnson-Mehl typical cell $\mathcal{C}$. Consider

$$\varphi(t) = \sum_{n \geq 1} \exp(-t\lambda_n), t > 0,$$

its spectral function. Besides, consider a Brownian bridge

$$(W_t(s))_{0 \leq s \leq t}, \quad W_t(0) \equiv W_t(t) \equiv 0, t \geq 0,$$

independent of the Johnson-Mehl tessellation. Using the notations of the section, we will see that the expectation of the spectral function $\varphi_{\mathcal{C}}(t)$ is connected with the "geometry" of the set $\mathbf{W}(2t)$. The probability associated with the Brownian bridge is still noted $\overline{\mathbf{P}}$.

Let us recall first (see [3], [15]) that the spectral function $\varphi_D(t)$, $t > 0$, of any bounded domain $D \subset \mathbb{R}^d$ can be expressed by means of the Brownian bridge under the form:

$$\varphi_D(t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_D \overline{\mathbf{P}}\{x + \sqrt{2t}\mathbf{W} \subset D\} dx. \tag{24}$$

(24) implies in particular the inequality

$$\varphi_D(t) \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} V_d(D). \tag{25}$$

Theorem 1 *The following identity holds:*

$$\mathbf{E}\varphi(t) = \frac{1}{\lambda(4\pi t)^{d/2}} \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \exp\left(- \int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right) \right\} dx\Lambda(du).$$

Proof. Applying (24) to the typical cell $\mathcal{C}$ and using Proposition 1, we obtain:

$$\begin{aligned} \mathbf{E}\varphi(t) &= \frac{1}{(4\pi t)^{d/2}} \overline{\mathbf{E}} \int \mathbf{P}\{x + \sqrt{2t}\widehat{\mathbf{W}} \subset \mathcal{C}\} dx \\ &= \frac{1}{\lambda(4\pi t)^{d/2}} \overline{\mathbf{E}} \int\int_0^\infty \exp\left\{ - \int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right\} \Lambda(du)dx, \end{aligned}$$

which completes the proof of theorem 1.

□

Theorem 1 and the estimations of subsection 1.3 provide the second-order asymptotic expansion of the function $\mathbf{E}\varphi_{\mathcal{C}}(t)$, $t > 0$, when $t \longrightarrow 0^+$:

Theorem 2 *If* Λ *satisfies the two conditions*

$$\textbf{(C1)} \int_0^\infty \left\{ \int_0^t \int_0^t [(t-u)^{3d-3} + (t-u)^{d-2}](1 + |u-w|^{2d-2})\Lambda(dw)\Lambda(du) \right\} \Lambda([0,t])p(t)dt < \infty,$$

$$\textbf{(C2)} \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \Lambda\left([||x|| + u, ||x|| + u + M] \right)^2 M^{2d} \right\} p(||x|| + u)dx\Lambda(du) < \infty,$$

then we have the following expansion when $t \longrightarrow 0^+$*:*

$$\mathbf{E}\varphi(t) = \frac{\mathbf{E}(V_d(\mathcal{C}))}{(4\pi t)^{\frac{d}{2}}} - \frac{\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})}{4\pi^{\frac{d}{2}}(d-1)!t^{\frac{d-1}{2}}} \mathbf{E}(V_{d-1}(\mathcal{C})) + O(\frac{1}{t^{\frac{d}{2}-1}}). \tag{26}$$

Proof. First let us establish the following intermediate lemma:

Lemma 4 *If* Λ *satisfies* **(C1)** *and* **(C2)***, then we have in the neighborhood of the origin:*

$$\begin{aligned} &\lambda(4\pi t)^{\frac{d}{2}}\mathbf{E}\varphi(t) \\ &\quad = \overline{\mathbf{E}} \left\{ \int_0^\infty \int_{\mathbb{R}^d} \left(1 - \left(\int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right.\right.\right. \\ &\qquad \left.\left.\left. - \int_0^{||x||+u} \omega_d(||x|| + u - w)^d \Lambda(dw) \right) \right) p(||x|| + u)dx\Lambda(du) \right\} + O(t). \end{aligned}$$

Proof of lemma 4. Write first $\mathbf{E}\varphi(t)$, $t > 0$, on the following lines

$$\begin{aligned} \lambda(4\pi t)^{\frac{d}{2}}\mathbf{E}\varphi(t) = \overline{\mathbf{E}} \int_0^\infty \int \exp &\left\{ \int_0^\infty V_d(\sqrt{2t}\mathbf{W}, x, u-w)\Lambda(dw) \right. \\ &\left. - \int_0^{||x||+u} \omega_d(||x|| + u - w)^d \Lambda(dw) \right\} p(||x|| + u)dx\Lambda(du), \end{aligned}$$

and let us prove that we can replace the exponential term above by the two first terms of its expansion.

Let us consider

$$\mathcal{D}(2t,x,u) = \int_0^\infty V_d(\mathbf{W}(2t),x,u-w)\Lambda(dw) - \int_0^{||x||+u} \omega_d(||x||+u-w)^d\Lambda(dw), \quad t>0, x\in\mathbb{R}^d, u\geq 0.$$

Then, using Taylor's formula with integral rest, we obtain that

$$\exp\{-\mathcal{D}(2t,x,u))\} - 1 + \mathcal{D}(2t,x,u) \leq \frac{1}{2}[\mathcal{D}(2t,x,u)]^2.$$

As a consequence, it suffices to show that the following integrals,

$$\mathcal{I}_1 = \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}}\left\{\left(\int_{||x||+u}^\infty V_d(\mathbf{W}(2t),x,u-w)\Lambda(dw)\right)^2\right\} p(||x||+u)dx\Lambda(du),$$

$$\mathcal{I}_2 = \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}}\left\{\left(\int_0^{||x||+u} |V_d(\mathbf{W}(2t),x,u-w) - V_d(\widehat{\mathbf{W}}(2t),f^{u-w}(x))|\Lambda(dw)\right)^2\right\} p(||x||+u)dx\Lambda(du),$$

$$\mathcal{I}_3 = \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}}\left\{\left(\int_0^{||x||+u} (V_d(2t,f^{u-w}(x)) - \omega_d(||x||+u-w)^d)\Lambda(dw)\right)^2\right\} p(||x||+u)dx\Lambda(du),$$

are $O(t)$ when $t \longrightarrow 0^+$. Let us focus on the first integral $\mathcal{I}_1$:

$$\begin{aligned}\int_{||x||+u}^\infty V_d(\mathbf{W}(2t),x,u-w)\Lambda(dw) &= \int_{||x||+u}^{||x||+u+M(2t)} V_d(\mathbf{W}(2t),x,u-w)\Lambda(dw) \\ &\leq V_d(\mathbf{W}(2t),x,-||x||)\Lambda([||x||+u,||x||+u+M(2t)]) \\ &\leq \omega_d(2M(2t))^d\Lambda([||x||+u,||x||+u+M(2t)]), \qquad (27)\end{aligned}$$

where $M(2t)$, $t>0$, is the maximum of the radial part of the Brownian bridge on the interval $0\leq s\leq 2t$. So, using (27), and the identity in law $M(2t) \stackrel{law}{=} \sqrt{2t}M(1) = \sqrt{2t}M$, we obtain for $t\leq \frac{1}{2}$ that

$$\mathcal{I}_1 \leq (2t)^d \int_0^\infty \int_{\mathbb{R}^d} 4^d{\omega_d}^2 \overline{\mathbf{E}}\left\{\Lambda\left([||x||+u,||x||+u+M(1)]\right)^2 M(1)^{2d}\right\} p(|x||+u)dx\Lambda(du).$$

Therefore, using the condition **(C2)**, we have that

$$\mathcal{I}_1 = O(t^d), \quad t\longrightarrow 0^+. \qquad (28)$$

Now we take into consideration $\mathcal{I}_2$. By using successively the Cauchy-Schwarz inequality and Lemma 3, we obtain:

$$\begin{aligned}\mathcal{I}_2 &\leq \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \int_0^{||x||+u} |V_d(\mathbf{W}(2t), x, u-w) - V_d(\widehat{\mathbf{W}}(2t), f^{u-w}(x))|^2 \Lambda(dw) \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Lambda([0, ||x|| + u]) p(||x|| + u) dx \Lambda(du) \\ &\leq tA(\int_0^\infty \int_{\mathbb{R}^d} (||x||^{2d-4} + 1) \left(\int_0^{||x||+u} (1 + |u-w|^{2d-2}) \Lambda(dw) \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Lambda([0, ||x|| + u]) p(||x|| + u) dx \Lambda(du).\end{aligned}$$

where $0 < A < \infty$ is a constant.
Thus since Λ satisfies **(C1)**, we deduce from the preceding inequality that

$$\mathcal{I}_2 = O(t), \quad t \longrightarrow 0^+. \tag{29}$$

Besides, we treat likewise the third integral $\mathcal{I}_3$ by using first the Cauchy-Schwarz inequality, then (21) which specifies the deviation $\left[V_d(\widehat{\mathbf{W}}(2t), f^{u-w}(x)) - \omega_d(||x|| + u - w)^d \right]$ in quadratic mean, and finally the fact that Λ satisfies condition **(C1)**. Thus,

$$\mathcal{I}_3 = O(t), \quad t \longrightarrow 0^+. \tag{30}$$

In conclusion, (28), (29) and (30) imply the statement of Lemma 4.

□

Returning now to the proof of Theorem 2, we remark that (27) implies too that

$$\int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \int_{||x||+u}^\infty V_d(\mathbf{W}(2t), x, u-w) \Lambda(dw) \right\} p(||x|| + u) dx \Lambda(du) = O(t), \quad t \longrightarrow 0^+. \tag{31}$$

Thus, we deduce from (31) and Lemma 4 that, when $t \longrightarrow 0^+$,

$$\begin{aligned}&(4\pi t)^{\frac{d}{2}} \lambda \mathbf{E} \varphi(t) \\ &= 1 + \int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ - \int_0^{||x||+u} \left(V_d(\mathbf{W}(2t), x, u-w) - \omega_d(||x|| + u - w)^d \right) \Lambda(dw) \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad p(||x|| + u) dx \Lambda(du) + O(t).\end{aligned}$$

Now, using the same arguments as in the proof of (29) and Lemma 2, we obtain when $t \to 0^+$,

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \int_0^{||x||+u} |V_d(\mathbf{W}(2t), x, u-w) - V_d(\widehat{\mathbf{W}}(2t), f^{u-w}(x))| \Lambda(dw) \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad p(||x|| + u) dx \Lambda(du) = O(t),\end{aligned}$$

and finally, (20) and Proposition 2 (ii) imply that

$$\int_0^\infty \int_{\mathbb{R}^d} \overline{\mathbf{E}} \left\{ \int_0^{||x||+u} (V_d(\widehat{\mathbf{W}}(2t), f^{u-w}(x)) - \omega_d(||x|| + u - w)^d)\Lambda(dw) \right\} \\ p(||x|| + u)dx\Lambda(du) \\ = \frac{2^{d-2}\Gamma(\frac{d+1}{2})\Gamma(\frac{d}{2})}{(d-1)!}\sqrt{t}\lambda\mathbf{E}(V_{d-1}(\mathcal{C})) + O(t), \quad t \longrightarrow 0^+,$$

which completes the proof of Theorem 2.

□

Remark 1 The conditions **(C1)** and **(C2)** are both satisfied in the usual cases considered in section 1.2. In the case when $\Lambda \neq \delta_0$, the boundary of Johnson-Mehl cells is the union of parts of hyperboloids and admits discontinued tangent hyperplanes . For such domains there is no results (to our knowledge) related to the third term of the asymptotic of $\varphi_D(t)$, $t > 0$. Consequently it would be interesting to manage, at least for some particular measures $\Lambda \neq \delta_0$ and even in dimension $d = 2$, the three-order expansion.

Appendix

Proofs of Lemmas 2 and 3.

The case $s \geq 0$.

Let us introduce first some new notations. For $y \in \widehat{\mathbf{W}}(2t)$, $u \in \mathbb{S}^{d-1}$, and $x \in \mathbb{R}^d$, we consider:

(i) $H_{y,u}$ the hyperplane perpendicular to u containing y;

(ii) $\mathcal{E}(u) = \{y \in \widehat{\mathbf{W}}(2t); H_{y,u} \cap (f^s(x) + \mathbb{R}_+u) \neq \emptyset\}$;

(iii) $l(y,u) = \begin{cases} d(f^s(x), H_{y,u}) & \text{if } y \in \mathcal{E}(u) \\ 0 & \text{if not} \end{cases}$;

(iv) $p(y,u)$ the distance between $f^s(x)$ and the point of intersection of the ball $B(y, ||y - x|| + s)$ with the half-line $f^s(x) + \mathbb{R}_+u$;

(v) $r(y,u) = p(y,u) - 2l(y,u) \geq 0$;

(vi) $l_{max}(u) = \sup_{y\in\widehat{\mathbf{W}}(2t)} l(y,u)$;

(vii) $p_{max}(u) = \sup_{y\in\widehat{\mathbf{W}}(2t)} p(y,u)$.

Notice that the following inclusions are satisfied:

$$\bigcup_{y\in\mathbf{W}(2t)} B(y, \left\|y - f^s(x)\right\|) \subset \bigcup_{y\in\mathbf{W}(2t)} B(y, ||y - x|| + s) \subset \bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y, ||y - x|| + s). \quad (32)$$

We can determine the d-dimensional area of the set

$$\left[\bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y,||y-x||+s)\right]\setminus\left[\bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y,\left\|y-f^s(x)\right\|)\right],$$

by integrating (as in the case $s=0$ studied previously [5]) in spherical coordinates (taking $f^s(x)$ for the origin):

$$V_d\left[\bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y,||y-x||+s)\right]-V_d\left[\bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y,\left\|y-f^s(x)\right\|)\right]$$
$$=\int_{\mathbb{S}^{d-1}}\int_{2l_{max}(u)}^{p_{max}(u)} r^{d-1}drd\nu_d(u)\leq\int_{\mathbb{S}^{d-1}}\frac{1}{d}\sup_{y\in\widehat{\mathbf{W}}(2t)}\left(p(y,u)^d-(2l(y,u))^d\right)d\nu_d(u). \quad (33)$$

For $u\in\mathbb{S}^{d-1}$ and $y\in\mathcal{E}(u)$, we have:

$$\begin{aligned} r(y,u)^2+2l(y,u)r(y,u) &= (||y-x||+s)^2-||y-f^s(x)||^2 \\ &= 2||y-x||s(1-\cos\beta(y)), \end{aligned}$$

where $\beta(y)$ denotes the angle spanned by the vectors $\overrightarrow{xy}$ and $\overrightarrow{x0}$.

If $y\notin\mathcal{E}(u)$, we have also

$$r(y,u)^2+2l(y,-u)r(y,u)=2||y-x||s(1-\cos\beta(y)).$$

Remark that $\beta(y)$ satisfies the following inequality:

$$||x||\sin\beta(y)\leq||y||\leq M(2t), \quad (34)$$

where $M(2t)$ denotes the maximum of the radial part of the Brownian bridge.

Elsewhere, let $M_0(2t)$ be the maximum of the projection of the Brownian bridge $W_{2t}(s)$,
$0\leq s\leq 2t$, on the oriented line $\mathbb{R}\overrightarrow{0x}$ orientated by that vector. If $M_0(2t)\leq||x||$, then $\beta(y)\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and (34) implies that

$$(1-\cos\beta(y))\leq\sin^2\beta(y)\leq\frac{M(2t)^2}{||x||^2}.$$

In conclusion, for all $y\in\widehat{\mathbf{W}}(2t)$, we get

$$\begin{aligned} r(y,u)^2+2l(y,u)r(y,u) &\leq 1_{\{M(2t)\leq||x||\}}2(M(2t)+||x||)s\frac{M(2t)^2}{||x||^2}+1_{\{M(2t)>||x||\}}8M(2t)s \\ &\leq 1_{\{M(2t)\leq||x||\}}4\frac{M(2t)^2}{||x||}s+1_{\{M(2t)>||x||\}}8M(2t)s. \quad (35) \end{aligned}$$

Now using (32), (33) and (35), the equivalence in law $M(2t) = \sqrt{2t}M$, $M_0(2t) = \sqrt{2t}M_0$, the fact that the moments of M are finite, and finally the obvious estimation $l(y,u) \leq (||x|| + s + M(2t))$, it follows that

$$
\begin{aligned}
&\overline{\mathbf{E}}\{V_d(\mathbf{W}(2t), x, s) - V_d(\widehat{\mathbf{W}}(2t), f^s(x))\} \\
&\leq \int_{\mathbb{S}^{d-1}} \frac{1}{d}\overline{\mathbf{E}}\left\{\sup_{y\in\sqrt{2t}\widehat{\mathbf{W}}(1)}\left(\sum_{i=1}^{d}\tbinom{d}{i}\, r(y,u)^i(2l(y,u))^{d-i}\right)\right\} d\nu_d(u) \\
&\leq Kt\overline{\mathbf{E}}\left\{1_{\{\sqrt{2t}M\leq||x||\}}\left(M^2(||x||+s)^{d-2}\frac{s}{||x||} + \sum_{i=2}^{d}(2t)^{\frac{i}{2}-1}M^i(||x||+s)^{d-i}\left(\frac{s}{||x||}\right)^{\frac{i}{2}}\right)\right. \\
&\qquad\left. + 1_{\{\sqrt{2t}M>||x||\}}\left(\frac{(s+\sqrt{2t}M)^{d-2}sM}{\sqrt{2t}} + \sum_{i=2}^{d}\frac{(2t)^{\frac{i}{2}-1}(s+\sqrt{2t}M)^{d-i}(sM)^{\frac{i}{2}}}{\sqrt{2t}}\right)\right\} \\
&\leq tA\left(||x||^{d-3} + \frac{1}{||x||}\right)(s+s^{d-1}), \quad x\in\mathbb{R}^d,\ \ s>0,\ \ t\leq 1/2,
\end{aligned}
$$

where K and A are positive constants.

Consequently, Lemma 2 is proved for $s \geq 0$.

In the same way, we obtain the following inequality valid for all $t > 0$ chosen small enough:

$$
\begin{aligned}
&\overline{\mathbf{E}}\left\{\left[V_d(\mathbf{W}(2t), x, s) - V_d(\widehat{\mathbf{W}}(2t), f^s(x))\right]^2\right\} \\
&\leq K\overline{\mathbf{E}}\left\{1_{\{\sqrt{2t}M\leq||x||\}}\left(t^2M^4(||x||+s)^{2d-4}\frac{s^2}{||x||^2} + \sum_{i=2}^{d}t^iM^{2i}(||x||+s)^{2d-2i}\left(\frac{s}{||x||}\right)^{i}\right)\right. \\
&\qquad\left. + 1_{\{\sqrt{2t}M>||x||\}}\left(t(s+M)^{d-2}s^2M^2 + \sum_{i=2}^{d}t(s+M)^{2d-2i}(sM)^i\right)\right\} \\
&\leq tA(||x||^{2d-2}+1)(1+s^{2d-2}), \quad x\in\mathbb{R}^d,\ \ s>0,\ \ t\leq 1/2,
\end{aligned}
$$

where K and A are positive constants. Thus Lemma 3 follows.

The case $s < 0$ with $||x|| + s \geq 0$.

Note that we have

$$
\bigcup_{\substack{y\in\mathbf{W}(2t)\\ ||y-x||+s>0}} B(y, ||y-x||+s) \subset \bigcup_{y\in\widehat{\mathbf{W}}(2t)} B(y, ||y-f^s(x)||).
$$

Consider a vector $u \in \mathbb{S}^{d-1}$ such that $H_u \cap (f^s(x) + \mathbb{R}_+u) \neq \emptyset$. Then the hyperplane H_u contains at least one extreme point of $\widehat{\mathbf{W}}(2t)$. This fact implies (according to [14], corollary 18.3.1) that H_u contains at least one point of the Brownian path $\mathbf{W}(2t)$. Fix then a point $y_u \in H_u \cap \mathbf{W}(2t)$ and notice that

$$
B(y_u, ||y_u - x|| + s) \cap (f^s(x) + \mathbb{R}_+u) \neq \emptyset
$$

if and only if

$$(||y_u - x|| + s)^2 \geq ||y_u - f^s(x)||^2 - m(f^s(x), u)^2,$$

which means

$$m(f^s(x), u)^2 \geq 2|s|||y_u - x||(1 - \cos\beta(y_u)). \tag{36}$$

Moreover, if $B(y_u, ||y_u - x|| + s) \cap (f^s(x) + \mathbb{R}_+ u) \neq \emptyset$, then that intersection is a segment whose extremities are respectively at distance $r(u)$ and $2m(f^s(x), u) - r(u)$ from $f^s(x)$. When the intersection is empty, we will note $r(u) = m(f^s(x), u)$.

From the obvious inequality

$$V_d(\mathbf{W}(2t), x, s) \geq \int_{\mathbb{S}^+ \cup \mathcal{A}(\widehat{\mathbf{W}}(2t), f^s(x))} \int_{r(u)}^{2m(f^s(x),u)-r(u)} \rho^{d-1} d\rho d\nu_d(u)$$

we deduce that

$$\begin{aligned}
&V_d(\widehat{\mathbf{W}}(2t), f^s(x)) - V_d(\mathbf{W}(2t), x, s) \\
&\leq \int_{\mathbb{S}^+ \cup \mathcal{A}(\widehat{\mathbf{W}}(2t), f^s(x))} \left(\int_0^{r(u)} \rho^{d-1} d\rho + \int_{2m(f^s(x),u)-r(u)}^{2m(f^s(x),u)} \rho^{d-1} d\rho \right) d\nu_d(u) \\
&\leq \int_{\mathbb{S}^+ \cup \mathcal{A}(\widehat{\mathbf{W}}(2t), f^s(x))} \frac{1}{d} \left(r(u)^d + \sum_{i=1}^{d} \tbinom{d}{i} r(u)^i q(u)^{d-i} \right) d\nu_d(u),
\end{aligned} \tag{37}$$

where $q(u) = 2m(f^s(x), u) - r(u)$.

Let us estimate the length $r(u)$:

(α) If $B(y_u, ||y_u - x|| + s) \cap (f^s(x) + \mathbb{R}_+ u) \neq \emptyset$, we obtain (proceeding in the same way as in the case $s > 0$) that

$$\begin{aligned}
r(u)^2 + 2(q(u) - m(f^s(x), u))r(u) &= -r(u)^2 + 2m(f^s(x), u)r(u) \\
&= 2||y_u - x|||s|(1 - \cos\beta(y_u)),
\end{aligned}$$

which implies that

$$r(u)^2 + 2q(u)r(u) \leq 1_{\{M(2t) \leq ||x||\}} K|s| \frac{M(2t)^2}{||x||} + 1_{\{M(2t) > ||x||\}} K'|s| M(2t), \tag{38}$$

where K, K' are positive constants.

(β) If $B(y_u, ||y_u - x|| + s) \cap (f^s(x) + \mathbb{R}_+ u) = \emptyset$, then

$$m(f^s(x), u)^2 = r(u)^2 < 2(1 - \cos\beta(y_u))|s|||y_u - x||,$$

and consequently, the majoration (38) is still valid.

Now using (38), the inequality $q(u) \leq 2(||x|| + M(2t))$ and the equivalence in law due to the scaling-property of Brownian bridge, we infer from (37) that:

$$
\begin{aligned}
&\overline{\mathbf{E}}\{V_d(\widehat{\mathbf{W}}(2t), f^s(x)) - V_d(\mathbf{W}(2t), x, s)\} \\
&\leq Kt\overline{\mathbf{E}}\left\{1_{\{\sqrt{2t}M(1)\leq ||x||\}}\left(|s|M^2||x||^{d-3} + \sum_{i=2}^{d}(2t)^{\frac{i}{2}-1}|s|^{i/2}M^i||x||^{d-\frac{3i}{2}}\right) + \right. \\
&\qquad\qquad \left. 1_{\{\sqrt{2t}M(1)>||x||\}}\left(|s|\frac{(2t)^{\frac{d-2}{2}}M^{d-1}}{\sqrt{2t}} + \sum_{i=2}^{d}|s|^{i/2}\frac{(2t)^{\frac{d}{2}-\frac{i}{4}-\frac{1}{2}}M^{d-\frac{i}{2}}}{\sqrt{2t}}\right)\right\} \\
&\leq At\left(||x||^{d-3} + \frac{1}{||x||}\right)(|s| + |s|^{d-1}), \quad x \in \mathbb{R}^d,\ s < 0,\ ||x|| + s < 0,\ t \leq 1/2,
\end{aligned}
$$

where K and A are positive constants.

So Lemma 2 is proved for $s < 0$. The proof of Lemma 3 is analogous to that presented in the case $s \geq 0$.

□

References

[1] R. Cowan, S. N. Chiu, and L. Holst. A limit theorem for the replication time of a DNA molecule. *J. Appl. Probab.*, 32(2):296–303, 1995.

[2] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33:958–972, 1962.

[3] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1):57–83, 1996.

[4] A. Goldman and P. Calka. On the spectral function of the Johnson-Mehl and Voronoi tessellations. *Preprint 00-02 of LaPCS*, 2000.

[5] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):835–840, 2001.

[6] W. A. Johnson and R. F. Mehl. Reaction kinetics in processes of nucleation and growth. *Trans. Amer. Inst. Min. Engrs.*, 135:416–458, 1939.

[7] M. Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[8] J. Møller. Random tessellations in $\mathbb{R}^d$. *Adv. in Appl. Probab.*, 21(1):37–73, 1989.

[9] J. Møller. Random Johnson-Mehl tessellations. *Adv. in Appl. Probab.*, 24(4):814–844, 1992.

[10] J. Møller. *Lectures on random Voronoĭ tessellations.* Springer-Verlag, New York, 1994.

[11] J. Møller. Generation of Johnson-Mehl crystals and comparative analysis of models for random nucleation. *Adv. in Appl. Probab.*, 27(2):367–383, 1995.

[12] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations: concepts and applications of Voronoi diagrams.* John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[13] M. H. Protter. Can one hear the shape of a drum? revisited. *SIAM Rev.*, 29(2):185–197, 1987.

[14] R. T. Rockafellar. *Convex analysis.* Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 28.

[15] D. W. Stroock. *Probability theory, an analytic view.* Cambridge University Press, Cambridge, 1993.

[16] M. van den Berg and S. Srisatkunarajah. Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary. *J. London Math. Soc. (2)*, 37(1):119–127, 1988.

7.3 A new calculation of the moments of the unoccupied part of a circle by a random covering of i.i.d. intervals of fixed length.

Let us consider for a fixed $p \in \mathbb{N}^*$, p i.i.d random variables $U_1, \cdots, U_p$ p uniformly distributed on the circle of circumference one. Considering a given symmetric relation $\mathcal{R}$ on the set $\{1, \cdots, p\}$ (or equivalently a graph Γ on a set with p vertices), we want to calculate the probability

$$f_r(\mathcal{R}) = f_r(\Gamma) = \mathbf{P}\{|U_i - U_j| > r \ \forall i, j \text{ such that } i\mathcal{R}j\}, \quad r > 0,$$

where $|x - y|$ denotes the arc length between two points x and y of the circle C.

The following lemma provides some elementary properties of the function f_r.

Lemma 7.3.1 **(i)** *f_r is a decreasing function for the inclusion of the symmetric relations ;*
(ii) *Let $\Gamma = \bigsqcup_{i=1}^{l_\Gamma} \Gamma_i$ be the decompostion in connex components of the graph Γ. Then*

$$f_r(\Gamma) = \prod_{i=1}^{l_\Gamma} f_r(\Gamma_i);$$

(iii) *Let Γ_0 be the graph connecting any couple of points (i, j), $i \neq j$. Then*

$$f_r(\Gamma_0) = (1 - pr)_+^{p-1}. \tag{7.7}$$

Proof. The points (i) and (ii) are clear. It remains to demonstrate the point (iii).

To this end, we fix a reference point u_0 on the circle C and we consider the random permutation Θ (which is unique almost surely) such that $U_{\Theta(1)}, \cdots, U_{\Theta(p)}$ are in trigonometric order from u_0. We denote by L_i, $1 \leq i \leq p$, the arc length between $U_{\Theta(i+1)}$ and $U_{\Theta(i)}$ (with the convention $\Theta(p+1) = \Theta(1)$).

Since the distribution of the vector $(U_1, \cdots, U_p)$ is exchangeable and invariant by translation on the circle, Θ is uniformly distributed on the set $\mathcal{S}_p$ of the permutations of p elements and the distribution of $(L_1, \cdots, L_p)$ is the (normalized) uniform measure σ_p on the simplex

$$\{(l_1, \cdots, l_p) \in [0,1]^p; \sum_{i=1}^p l_i = 1\}.$$

Consequently, we have

$$\begin{aligned} f_r(\Gamma_0) &= \mathbf{P}\{|U_i - U_j| > r \ \forall i \neq j\} \\ &= \mathbf{P}\{L_i > r, 1 \leq i \leq p\} \\ &= \int \prod_{i=1}^p \mathbf{1}_{(r,1]}(l_i) d\sigma_p(l_1, \cdots, l_p). \end{aligned}$$

We calculate the last expression by classical integration methods in order to obtain the formula (7.7).

$\square$

The following proposition provides in some cases the explicit calculation of the function f_r.

Proposition 7.3.2 *Let Γ be a graph such that two connected points cannot be connected by a path without edges. Then*

$$f_r(\Gamma) = \frac{1}{p!} \sum_{\tau \in \mathcal{S}_p} (1 - n_\Gamma(\tau) r)_+^{p-1},$$

where $n_\Gamma(\tau)$ denotes the number of edges we meet when we follow the circuit $\tau(1)$, $\tau(2)$, $\cdots$, $\tau(p)$, $\tau(1)$.

Proof. Θ and $L_1, \cdots, L_p$ being as in the preceding proof, we have

$$\begin{aligned} f_r(\Gamma) &= \sum_{\tau \in \mathcal{S}_p} \mathbf{P}\{|U_i - U_j| > r \ \forall i, j \text{ such that } i \mathcal{R} j; \Theta = \tau\} \\ &= \frac{1}{p!} \sum_{\tau \in \mathcal{S}_p} \mathbf{P}\{|U_i - U_j| > r \ \forall i, j \text{ such that } i \mathcal{R} j | \Theta = \tau\} \end{aligned} \tag{7.8}$$

We have to calculate for each fixed $\tau \in \mathcal{S}_p$ the probability

$$\mathbf{P}\{|U_i - U_j| > r \ \forall i, j \text{ such that } i \mathcal{R} j | \Theta = \tau\}.$$

Let us denote by $I_\tau(\mathcal{R})$ the set $\{1 \leq i \leq p;\ \tau(i)\ \mathcal{R}\ \tau(i+1)\}$ and $n_\Gamma(\tau) = \# I_\tau(\mathcal{R})$. We then have

$$\mathbf{P}\{|U_i - U_j| > r \ \forall i, j \text{ such that } i \mathcal{R} j | \Theta = \tau\} = \mathbf{P}\{L_i > r \ \forall i \in I_\tau(\mathcal{R})\}. \tag{7.9}$$

Indeed, conditionally to $\{\Theta = \tau\}$, if $|U_i - U_j| > r$ for all i, j such that $i \mathcal{R} j$, in particular we have $|U_{\tau(i+1)} - U_{\tau(i)}| = L_i > r$ for all i in the set $I_\tau(\mathcal{R})$.

Conversely let us suppose that $L_i > r$ for all $i \in I_\tau(\mathcal{R})$ and consider a couple of integers (i, j) such that $i = \tau(k)$, $j = \tau(l)$, $k < l$, and $i \mathcal{R} j$.

There then exists $m \in [k, l)$ such that $m \in I_\tau(\mathcal{R})$ because otherwise $\tau(k), \tau(k+1), \cdots, \tau(l)$ would be a path without edge between i and j, which is excluded. Consequently since $L_m > r$, a fortiori $|U_i - U_j| > r$. The equality (7.9) then is proved.

It remains to determine the probability $\mathbf{P}\{L_i > r \ \forall i \in I_\tau(\mathcal{R})\}$. The distribution of the vector $(L_1, \cdots, L_p)$ being the measure σ_p, we obtain after some calculation that

$$\begin{aligned} \mathbf{P}\{L_i > r \ \forall i \in I_\tau(\mathcal{R})\} &= \int \prod_{i \in I_\tau(\mathcal{R})} \mathbf{1}_{(r,1]}(l_i) d\sigma_p(l_1, \cdots, l_p) \\ &= (1 - \# I_\tau(\mathcal{R}) \times r)^{p-1} \\ &= (1 - n_\Gamma(\tau) r)^{p-1}. \end{aligned} \tag{7.10}$$

Inserting the result (7.10) in the equality (7.8), we deduce Proposition 7.3.2.

$\square$

Let us now consider the following problem : we place at each point U_i, $1 \le i \le p$, an interval of the circle $\mathcal{A}(U_i)$ centered at U_i and of length $2r$. We want to determine the different moments of the Lebesgue measure $L(U_1, \cdots, U_p)$ of the unoccupied part of the circle. These calculations have already been completed by Siegel [80]. We here provide in the next theorem a new way to obtain the results of Siegel by using Proposition 7.3.2.

Theorem 7.3.3 *For every $n \in \mathbb{N}$, we have*

$$\mathbf{E}\left(L(U_1, \cdots, U_p)^n\right) = \binom{n+p-1}{p}^{-1} \sum_{k=1}^{n} \binom{n}{k}\binom{p-1}{k-1}(1-2kr)_+^{n+p-1},$$

with the convention $\binom{r}{s} = 0$ if $s > r$.

Proof. Let us consider a sequence $\{V_i; i \ge 1\}$ of i.i.d. variables uniformly distributed on the circle, and independent of the sequence $\{U_i; 1 \le i \le p\}$.

Using Fubini's theorem, we can rewrite the n-th moment of $L(U_1, \cdots, U_p)$ in the following way.

$$\begin{aligned}
\mathbf{E}\left(L(U_1, \cdots, U_p)^n\right) &= \mathbf{P}\{V_1, \cdots, V_n \notin \cup_{i=1}^p \mathcal{A}(U_i)\} \\
&= \mathbf{P}\{|V_j - U_i| > r \forall i, j; 1 \le i \le p, 1 \le j \le n\} \\
&= f_r(\Lambda_{n,p}), \qquad (7.11)
\end{aligned}$$

where $\Lambda_{n,p}$ is the graph with $n+p$ vertices separated into two parts Λ_1 and Λ_2 such that $\#\Lambda_1 = n$ and $\#\Lambda_2 = p$, every vertex of Λ_1 is connected by an edge to every vertex of Λ_2 and there is no connection between two vertices of a same part.

Applying then Proposition 7.3.2 whose hypothesis is clearly satisfied by $\Lambda_{n,p}$, the equality (7.11) implies that

$$\begin{aligned}
\mathbf{E}\left(L(U_1, \cdots, U_p)^n\right) &= \frac{1}{(n+p)!} \sum_{\tau \in \mathcal{S}_{n+p}} (1 - n_{\Lambda_{n,p}}(\tau) r)_+^{n+p-1} \\
&= \frac{1}{(n+p)!} \sum_{l \ge 0} N_{n,p}(l)(1 - lr)_+^{n+p-1}, \qquad (7.12)
\end{aligned}$$

where $N_{n,p}(l)$, $l \ge 0$ denotes the number of paths which contain exactly l edges.

Let us remark that the definition of $\Lambda_{n,p}$ implies that the number of edges that a path crosses is exactly the double of the number of travellings from from Λ_1 to Λ_2. Consequently, $N_{n,p}(l)$ is different from zero if l is even and $2 \le l \le 2\min(n,p)$.

For every $1 \le k \le \min(n,p)$. let us denote $M_m(k)$ (respectively $M'_m(k)$), $m \ge k$, the number of ways to sort m elements and divide the sorted elements into k classes, where we suppose the first element can (respectively cannot) be in the same class as the last.

We then have the equality

$$N_{n,p}(2k) = M_n(k) \times M'_p(k) + M'_n(k) \times M_p(k). \qquad (7.13)$$

In addition, we can determine the expression of $M_m(k)$ (respectively $M'_m(k)$), $m \geq k \geq 1$:

$$M_m(k) = m!\binom{m}{k} \text{ and } M'_m(k) = m!\binom{m-1}{k-1} \tag{7.14}$$

Inserting the result (7.14) in the equation (7.13), we then obtain

$$N_{n,p}(2k) = \frac{n!(n-1)!p!(p-1)!}{k!(k-1)!(n-k)!(p-k)!}(n+p).$$

Using (7.12), we deduce that

$$\begin{aligned} \mathbf{E}\left(L(U_1, \cdots, U_p)^n\right) &= \frac{1}{(n+p-1))!} \sum_{k=1}^{\min(n,p)} \frac{n!(n-1)!p!(p-1)!}{k!(k-1)!(n-k)!(p-k)!}(1-2kr)_+^{n+p-1} \\ &= \binom{n+p-1}{p}^{-1} \sum_{k=1}^{n} \binom{n}{k}\binom{p-1}{k-1}(1-2kr)_+^{n+p-1} \end{aligned}$$

This completes the proof of Theorem 7.3.3.

□

Problèmes ouverts.

Chapitre 2.

2.2. et **2.3.** (i) Les résultats de ces parties concernent uniquement des mosaïques deux-dimensionnelles. Cependant, les propositions fondamentales (Propositions 1 et 2 du second article) se généralisent aisément aux dimensions supérieures. Dans le cas de la mosaïque poissonienne d'hyperplans, le rôle qu'avait le périmètre du polygône dans la formule (22) est tenu en dimension $n \geq 3$, par le diamètre moyen d'un polyèdre. La difficulté pour expliciter les différentes lois est donc d'ordre purement géométrique : pour ce qui est de la mosaïque de Poisson-Voronoi, il faut savoir exprimer en fonction des coordonnées sphériques des voisins de l'origine le volume du domaine fondamental associé au polyèdre formé par les hyperplans médiateurs entre l'origine et ses voisins. Pour cela, nous disposons de la formule théorique fournie par la proposition 2 de l'article 3.2 qui est valable en toute dimension. Le grand nombre de paramètres (toutes les coordonnées sphériques des voisins), rend ardu le calcul intégral. Il en va de même dans le cas de la mosaïque poissonienne d'hyperplans pour exprimer explicitement le diamètre moyen du polyèdre formé par tous les hyperplans polaires qui bordent la cellule de Crofton.
(ii) Les formules obtenues en dimension deux sont exactes et permettent le calcul numérique approché aussi précis que désiré. Elles sont cependant peu maniables, plus particulièrement l'expression des lois de l'aire et du périmètre. Il paraît difficile par exemple de retrouver à partir de ces formules les valeurs des moments connus par ailleurs. Les résultats de l'article 2.3 ne permettent pas non plus *a priori* d'obtenir une estimation asymptotique précise des queues des lois.
(iii) Rappelons que A. Goldman a obtenu par des techniques tout à fait différentes un équivalent logarithmique de la queue de l'aire de la cellule typique (resp. de la cellule de Crofton) d'une mosaïque poissonienne de droites. Il serait intéressant d'avoir un résultat équivalent pour la cellule typique de Poisson-Voronoi. En effet, on pourrait ainsi comparer les lois des aires de la cellule $\mathcal{C}$ et du plus grand disque inscrit dans $\mathcal{C}$ et en déduire une forme équivalente de la conjecture énoncée par D. G. Kendall dans le cas des mosaïques poissoniennes de droites : lorsque l'aire de la cellule typique est "grande", sa forme est "approximativement circulaire".

Toujours dans le cadre de cette conjecture, il serait intéressant de prouver que le nombre de côtés de la cellule typique "croît" avec son volume et de fournir une estimation de sa vitesse de croissance.

2.4. (i) De même, on peut se demander s'il serait possible de prolonger les résultats obtenus sur la loi conjointe des rayons R_m et R_M en dimension supérieure. En effet, la liaison fondamentale fournie par le théorème 2 du paragraphe 2.4 entre la fonction de répartition de R_M et le recouvrement du cercle par des arcs aléatoires indépendants et identiquement distribués peut se généraliser en conservant les mêmes arguments de démonstration. En dimension $d \geq 3$, il faut alors considérer le recouvrement de la sphère $(d-1)$-dimensionnelle par des calottes i.i.d. dont les centres sont de loi uniforme sur la sphère et dont les rayons suivent une loi que l'on peut expliciter. Cependant, on ne dispose pas de beaucoup de résultats sur ce nouveau modèle de recouvrement. Le cas où les calottes ont des rayons constants et tous égaux a été abordé notamment par P. Hall [32], mais nous n'avons aucun moyen de comparaison avec les recouvrements par des calottes de rayons aléatoires.
(ii) Par ailleurs, notons R_I (resp. R_C) le rayon du plus grand (resp. petit) disque inclus dans (resp. contenant) la cellule typique $\mathcal{C}$. En d'autres termes, R_I et R_C sont les vrais rayons de disque inscrit et circonscrit de $\mathcal{C}$. Leur définition est intrinsèque à la cellule typique et ne dépend pas de la réalisation $C(0)$, contrairement aux rayons R_m et R_M. On a trivialement les inégalités

$$R_m \leq R_I \leq R_C \leq R_M,$$

mais la question de la détermination de la loi de R_I ou R_C reste ouverte. On sait tout de même en utilisant le résultat asymptotique donné par le théorème 5 que pour tout $-1 < \alpha < \frac{1}{3}$,

$$\lim_{r\to+\infty} \mathbf{P}\{R_C \geq r + \frac{1}{r^\alpha} | R_m = r\} = \lim_{r\to+\infty} \mathbf{P}\{R_M \geq r + \frac{1}{r^\alpha} | R_m = r\} = 0.$$

(iii) Enfin, le théorème 5 exprime l'idée qu'avec une "forte probabilité", quand le rayon R_m est "grand", la frontière de la cellule typique se trouve dans une couronne d'épaisseur de l'ordre de $R_m^{-1/3}$. Cependant, cela ne résout pas totalement la conjecture de D. G. Kendall citée plus haut. En effet, il peut arriver que la cellule $C(0)$ soit de forme très effilée, c'est-à-dire avec une aire grande mais un rayon R_m petit. Il faudrait donc justifier rigoureusement que l'on se trouve dans de tels cas avec une probabilité "faible".

Chapitre 3.

Les problèmes ouverts concernant le spectre du Laplacien de la cellule typique de Poisson-Voronoi sont présentés dans la dernière partie de l'article 3.2. Principalement, il reste à :
(i) comprendre la signification géométrique du troisième terme du développement de l'espérance de la fonction spectrale au voisinage de l'origine pour les dimensions supérieures à trois ;
(ii) tenter de poursuivre le développement en calculant certaines covariances associées au pont brownien ;
(iii) prolonger le résultat asymptotique sur la transformée de Laplace de la première valeur propre en dimension supérieure à deux.

Comme il est dit dans l'article (partie 5, (iv)), il suffirait pour obtenir le point (iii) ci-dessus de disposer d'un équivalent logarithmique de la fonction de répartition du diamètre moyen de l'enveloppe convexe de la trajectoire du pont brownien. Une autre méthode serait d'estimer la transformée de Laplace du volume de cette enveloppe convexe en la comparant à la saucisse de Wiener pour laquelle des résultats asymptotiques précis ont été obtenus [19], [88].

Chapitre 6.

(i) Le modèle de fissuration proposé est unidirectionnel. Il serait intéressant de proposer une généralisation au moins deux-dimensionnelle de la construction des positions des fissures qui tiendrait compte de la relaxation de contrainte. On peut songer au réseau aléatoire, proposé par M. Schlather et D. Stoyan [78], obtenu en réduisant les arêtes d'une mosaïque de Poisson-Voronoi.
(ii) La zone de relaxation choisie est constante dans notre travail. Il serait plus cohérent de la prendre aléatoire pour chaque fissure, d'autant plus que cette zone dépend également de la longueur de la fissure elle-même sur le dépôt (rappelons que dans notre modèle, nous avons supposé les fissures toutes parallèles et de longueur infinie). Il serait donc intéressant d'associer à chaque position de fissure "potentielle" une distance de relaxation aléatoire suivant une loi fixée. Dans ce contexte, la construction du processus stationnaire des fissures sur la droite réelle avec hypothèse de relaxation de contrainte est toujours possible, de même que l'application de la formule de Slivnyak conduisant à une expression simplifiée de l'intensité du processus et de la loi de la distance inter-fissures typique (voir la proposition 4 de l'article). La généralisation de l'équation fonctionnelle est plus difficile et on ne peut donc pas dire s'il est envisageable d'obtenir des formules explicites comme dans le cas où la zone de relaxation est de longueur constante.

On peut aussi songer à adopter un modèle dynamique dans lequel cet intervalle grossirait progressivement avec le temps jusqu'à atteindre une valeur limite. Les mosaïques de Johnson-Mehl unidimensionnelles pourraient jouer un rôle dans ce contexte.
(iii) Certaines questions restent ouvertes sur notre modèle proprement dit. Par exemple, on ne sait toujours pas déterminer, à niveau de contrainte fixé ou à saturation, la loi (ni même la moyenne) du nombre de fissures dans l'intervalle $[0.L]$. Par ailleurs, le calcul de la loi de la première fissure à droite de l'origine dans le modèle bilatéral (c'est-à-dire $\inf\{\Lambda_\varepsilon \cap \mathbb{R}_+\}$, $\varepsilon \geq 0$, n'a pour l'instant pas abouti.

Autres problèmes (non abordés dans cette thèse).

La mosaïque de Poisson-Voronoi est utilisée classiquement pour modéliser les matériaux biphasés et étudier leurs propriétés physiques par des techniques de simulation [40]. Il serait intéressant dans ce contexte de disposer d'un théorie de la conductivité thermique. Ce problème est lié à l'étude de la trajectoire d'un mouvement brownien dans la mosaïque de Poisson-Voronoi, indépendant de cette mosaïque et "changeant de vitesse" lorsqu'il traverse une frontière. On pourrait également songer à exploiter le comportement (récurrence ou transience) d'une marche au hasard simple sur l'ensemble des germes, c'est-à-dire la mosaïque duale de Delaunay.

Liste des publications.

Articles publiés ou à paraître.

1. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles.
C. R. Acad. Sci. Paris, Série I Math., 332 (6), 557-562 (2001)

2. Sur la fonction spectrale des cellules de Poisson-Voronoi.
En collaboration avec André Goldman.
C. R. Acad. Sci. Paris, Série I Math., 332 (9), 835-840 (2001)

3. La loi du plus petit disque contenant la cellule typique de Poisson-Voronoi.
C. R. Acad. Sci. Paris, Série I Math., 334 (4), 325-330 (2002)

4. The distributions of the smallest disk containing the Poisson-Voronoi typical cell and the Crofton cell in the plane.
A paraître dans *Adv. in Appl. Probab.* (2002)

Soumissions.

5. On the spectral function of the Poisson-Voronoi cells.
En collaboration avec André Goldman.

6. Poissonian tessellations of the Euclidean space. An extension of a result of R. E. Miles.

7. A rigorous proof of a result of R. E. Miles concerning the thickened Poisson hyperplane process in $\mathbb{R}^d$.

8. The explicit expression of the distribution of the number of sides of the typical Poisson-Voronoi cell.

9. Precise formulas for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process.

10. Stochastic modelling of a unidirectional multicracking phenomenon.
En collaboration avec André Mézin et Pierre Vallois.

Manuscrits en préparation.

11. On the spectral function of the Johnson-Mehl cells.
en collaboration avec André Goldman.

Bibliographie

[1] T. W. Anderson. The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.*, 6 :170–176, 1955.

[2] F. Baccelli, M. Klein, M. Lebourges, and S. Zuyev. Géométrie aléatoire et architecture de réseaux de communications. *Annales des Télécommunications*, 51 :158–179, 1996.

[3] F. Baccelli, K. Tchoumatchenko, and S. Zuyev. Markov paths on the Poisson-Delaunay graph with applications to routeing in mobile networks. *Adv. in Appl. Probab.*, 32(1) :1–18, 2000.

[4] F. Baccelli and S. Zuyev. Stochastic geometry models of mobile communication networks. In *Frontiers in queueing*, pages 227–243. CRC, Boca Raton, FL, 1997.

[5] J. Berréhar, C. Lapersonne-Meyer, M. Schott, and J. Villain. Formation of periodic crack structures in polydiacetylene single crystal thin films. *J. de Physique France*, 50 :923–935, 1989.

[6] P. Calka. Mosaïques poissoniennes de l'espace euclidien. Une extension d'un résultat de R. E. Miles. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(6) :557–562, 2001.

[7] P. Calka. A rigorous proof of a result of r. e. miles concerning the thickened poisson hyperplane process in $\mathbb{R}^d$. Preprint, 2001.

[8] P. Calka. The distributions of the smallest disks containing the Poisson-Voronoi typical cell and the Crofton cell in the plane. *To appear in Adv. in Appl. Probab.*, 2002.

[9] P. Calka. The explicit expression of the distribution of the number of sides of the typical Poisson-Voronoi cell. Preprint of LaPCS, 02-02, 2002.

[10] P. Calka. Precise formulas for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional poisson-voronoi tessellation and a poisson line process. Preprint, 2002.

[11] A. K. Chakravarti and O. W. Archibold. Patterns of diurnal variation of growing season precipitation on the Canadian prairies : a harmonic analysis. *The Canandian Geographer*, 37(1) :16–28, 1995.

[12] J. M. Chassery and M. Melkemi. Diagramme de voronoi appliqués à la segmentation d'images et à la détection d'évènements en imagerie multi-sources. *Traitement du signal*, 8(3) :155–164, 1991.

[13] S. N. Chiu, R. van de Weygaert, and D. Stoyan. The sectional Poisson Voronoi tessellation is not a Voronoi tessellation. *Adv. in Appl. Probab.*, 28(2) :356–376, 1996.

[14] S. N. Chiu and C. C. Yin. The time of completion of a linear birth-growth model. *Adv. in Appl. Probab.*, 32(3) :620–627, 2000.

[15] R. Cowan. The use of the ergodic theorems in random geometry. *Adv. Appl. Probab.*, (suppl.) :47–57, 1978. Spatial patterns and processes (Proc. Conf., Canberra, 1977).

[16] R. Cowan. Properties of ergodic random mosaic processes. *Math. Nachr.*, 97 :89–102, 1980.

[17] R. Cowan, S. N. Chiu, and L. Holst. A limit theorem for the replication time of a DNA molecule. *J. Appl. Probab.*, 32(2) :296–303, 1995.

[18] G. L. Dirichlet. über die reduction der positiven quadratischen formen mit drei unbestimmten ganzen zahlen. *J. für die reine und angewandte Math.*, 40 :209–227, 1850.

[19] M. D. Donsker and S. R. S. Varadhan. Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.*, 28(4) :525–565, 1975.

[20] A. Dvoretzky and H. Robbins. On the "parking" problem. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9 :209–225, 1964.

[21] S. G. Foss and S. A. Zuyev. On a Voronoi aggregative process related to a bivariate Poisson process. *Adv. in Appl. Probab.*, 28(4) :965–981, 1996.

[22] M. Gerstein, J. Tsai, and M. Levitt. The volume of atoms on the protein surface : calculated from simulation, using Voronoi polyhedra. *J. Mol. Biol.*, 249 :955–966, 1995.

[23] E. N. Gilbert. Random subdivisions of space into crystals. *Ann. Math. Statist.*, 33 :958–972, 1962.

[24] E. N. Gilbert. Random plane networks and needle-shaped crystals. In *Applications of undergraduate Mathematics in Engineering.* Macmillan, 1967.

[25] G. Gille. Investigations on mechanical behaviour of brittle wear-resistant coatings : II Theory. *Thin Solid Films*, 111 :201–218, 1984.

[26] A. Goldman. Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convexe du pont brownien. *Probab. Theory Related Fields*, 105(1) :57–83, 1996.

[27] A. Goldman. Sur le caractère presque-circulaire des petites enveloppes convexes des trajectoires du mouvement brownien plan. *Preprint of LaPCS*, (98-01), 1998.

[28] A. Goldman. Sur une conjecture de D. G. Kendall concernant la cellule de Crofton du plan et sur sa contrepartie brownienne. *Ann. Probab.*, 26(4) :1727–1750, 1998.

[29] A. Goldman and P. Calka. On the spectral function of the Johnson-Mehl and Voronoi tessellations. *Preprint 00-02 of LaPCS*, 2000.

[30] A. Goldman and P. Calka. Sur la fonction spectrale des cellules de Poisson-Voronoi. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9) :835–840, 2001.

[31] S. Goudsmit. Random distribution of lines in a plane. *Rev. Modern Phys.*, 17 :321–322, 1945.

[32] P. Hall. *Introduction to the theory of coverage processes.* John Wiley & Sons Inc., New York, 1988.

[33] A. Hayen and M. Quine. The proportion of triangles in a Poisson-Voronoi tessellation of the plane. *Adv. in Appl. Probab.*, 32(1) :67–74, 2000.

[34] A. Hayen and M. P. Quine. Areas of components of a Voronoi polygon in a homogeneous Poisson process in the plane. *Adv. in Appl. Probab.*, 34(2) :281–291, 2002.

[35] L. Heinrich. Contact and chord length distribution of a stationary Voronoĭ tessellation. *Adv. in Appl. Probab.*, 30(3) :603–618, 1998.

[36] A. L. Hinde and R. E. Miles. Monte-Carlo estimates of the distributions of the random polygons of the voronoi tessellation with respect to a poisson process. *J. of Stat. Comput. and Simul.*, 10 :205–223, 1980.

[37] Y. Isokawa. Poisson-Voronoi tessellations in three-dimensional hyperbolic spaces. *Adv. in Appl. Probab.*, 32(3) :648–662, 2000.

[38] W. A. Johnson and R. F. Mehl. Reaction kinetics in processes of nucleation and growth. *Trans. Amer. Inst. Min. Engrs.*, 135 :416–458, 1939.

[39] J. P. Kahane. Recouvrements aléatoires. *Gaz. Math.*, (53) :115–129, 1992.

[40] M. Kobayashi, H. Maekawa, and Y. Kondou. Calculation of the mean thermal conductivity of a heterogeneous solid mixture with the Voronoi-Polyhedron element method. *Trans. JSME*, 57B(537) :1795–1801, 1991.

[41] I. N. Kovalenko. On certain random polygons of large areas. *J. Appl. Math. Stochastic Anal.*, 11(3) :369–376, 1998.

[42] I. N. Kovalenko. A simplified proof of a conjecture of D. G. Kendall concerning shapes of random polygons. *J. Appl. Math. Stochastic Anal.*, 12(4) :301–310, 1999.

[43] S. Kumar and S. K. Kurtz. A Monte-Carlo study of size and angular properties of a three dimensional Poisson-Delaunay cell. *J. Stat. Phys.*, 75 :735–748, 1993.

[44] G. Le Caër and J. S. Ho. The voronoi tessellation generated from eigenvalues of complex random matrices. *J. Phys. A : Math. Gen.*, 1990.

[45] Y. Leterrier, D. Pellaton, D. Mendels, R. Glauser, J. Andersons, and J. A. Manson. Biaxial fragmentation on thin silicon oxide coatings on poly(ethylene terephtalate). *J. Mat. Sci.*, 36 :2213–2225, 2001.

[46] D. Mannion. Random packing of an interval. *Advances in Appl. Probability*, 8(3) :477–501, 1976.

[47] B. Massé. On the LLN for the number of vertices of a random convex hull. *Adv. in Appl. Probab.*, 32(3) :675–681, 2000.

[48] G. Matheron. *Random sets and integral geometry.* John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[49] M. S. Meade, J. W. Florin, and W. M. Gesler. *Medical geography.* Guilford Press, New-York.

[50] J. L. Meijering. Interface area, edge length, and number of vertices in crystal aggregates with random nucleation. *Philips Res. Rep.*, 8, 1953.

[51] M. Melkemi and D. Vandorpe. Voronoi diagrams and applications. In *Eigth Canadian Conference on Image Processing and Pattern Recognition*, pages 88–95, Canada, 1994. CIPPR Society.Banff-Alberta.

[52] A. Mézin, J. Lepage, N. Pacia, and D. Paulmier. Étude statistique de la fissuration des revêtements I : Théorie. *Thin Solid Films*, 172 :197–209, 1989.

[53] A. Mézin and P. Vallois. Statistical analysis of unidirectional multicracking of coatings by a two-dimensional Poisson point process. *Math. Mech. Solids*, 5(4) :417–440, 2000.

[54] R. E. Miles. Random polygons determined by random lines in a plane. *Proc. Nat. Acad. Sci. U.S.A.*, 52 :901–907, 1964.

[55] R. E. Miles. Random polygons determined by random lines in a plane. II. *Proc. Nat. Acad. Sci. U.S.A.*, 52 :1157–1160, 1964.

[56] R. E. Miles. Poisson flats in Euclidean spaces. I. A finite number of random uniform flats. *Advances in Appl. Probability*, 1 :211–237, 1969.

[57] R. E. Miles. On the homogeneous planar Poisson point process. *Math. Biosci.*, 6 :85–127, 1970.

[58] R. E. Miles. The random division of space. *Suppl. Adv. Appl. Probab.*, pages 243–266, 1972.

[59] R. E. Miles. The various aggregates of random polygons determined by random lines in a plane. *Advances in Math.*, 10 :256–290, 1973.

[60] R. E. Miles and R. J. Maillardet. The basic structures of Voronoï and generalized Voronoï polygons. *J. Appl. Probab.*, (Special Vol. 19A) :97–111, 1982. Essays in statistical science.

[61] I. Molchanov and S. Zuyev. Variational analysis of functionals of Poisson processes. *Math. Oper. Res.*, 25(3) :485–508, 2000.

[62] I. S. Molchanov. *Limit theorems for unions of random closed sets.* Springer-Verlag, Berlin, 1993.

[63] J. Møller. Random tessellations in $\mathbb{R}^d$. *Adv. in Appl. Probab.*, 21(1) :37–73, 1989.

[64] J. Møller. Random Johnson-Mehl tessellations. *Adv. in Appl. Probab.*, 24(4) :814–844, 1992.

[65] J. Møller. *Lectures on random Voronoï tessellations.* Springer-Verlag, New York, 1994.

[66] L. Muche. The Poisson Voronoi tessellation. III. Miles' formula. *Math. Nachr.*, 191 :247–267, 1998.

[67] L. Muche and D. Stoyan. Contact and chord length distributions of the Poisson Voronoï tessellation. *J. Appl. Probab.*, 29(2) :467–471, 1992.

[68] P. E. Ney. A random interval filling problem. *Ann. Math. Statist.*, 33 :702–718, 1962.

[69] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial tessellations : concepts and applications of Voronoi diagrams.* John Wiley & Sons Ltd., Chichester, second edition, 2000. With a foreword by D. G. Kendall.

[70] C. Palm. Intensitätsschwankungen im Fernsprechverkehr. *Ericsson Technics no.*, 44 :189, 1943.

[71] K. Paroux. *Théorèmes centraux limites pour les processus poissoniens de droites dans le plan et questions de convergence pour le modèle booléen de l'espace euclidien.* PhD thesis, Univ. Lyon 1, 1997.

[72] K. Paroux. Quelques théorèmes centraux limites pour les processus Poissoniens de droites dans le plan. *Adv. in Appl. Probab.*, 30(3) :640–656, 1998.

[73] M. D. Penrose. Random parking, sequential adsorption, and the jamming limit. *Comm. Math. Phys.*, 218(1) :153–176, 2001.

[74] E. Pielou. *Mathematical ecology.* Wiley-Interscience, New-York, 1977.

[75] A. Rényi. On a one-dimensional problem concerning random space filling. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 3(no 1/2) :109–127, 1958.

[76] B. D. Ripley. *Spatial statistics.* John Wiley & Sons Inc., New York, 1981. Wiley Series in Probability and Mathematical Statistics.

[77] L. A. Santaló. *Integral geometry and geometric probability.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

[78] M. Schlather and D. Stoyan. Edge systems of time-dependent incomplete Poisson Voronoi tessellations. *Comm. Statist. Stochastic Models*, 15(4) :599–615, 1999.

[79] L. A. Shepp. Covering the circle with random arcs. *Israel J. Math.*, 11 :328–345, 1972.

[80] A. F. Siegel. Random arcs on the circle. *J. Appl. Probab.*, 15(4) :774–789, 1978.

[81] A. F. Siegel. Random space filling and moments of coverage in geometrical probability. *J. Appl. Probab.*, 15(2) :340–355, 1978.

[82] A. F. Siegel and L. Holst. Covering the circle with random arcs of random sizes. *J. Appl. Probab.*, 19(2) :373–381, 1982.

[83] I. M. Slivnyak. Some properties of stationary flows of homogeneous random events. *Theor. Probab. Appl.*, 7 :336–341, 1962.

[84] H. Solomon. *Geometric probability.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1978. Ten lectures given at the University of Nevada, Las Vegas, Nev., June 9–13, 1975, Conference Board of the Mathematical Sciences—Regional Conference Series in Applied Mathematics, No. 28.

[85] W. L. Stevens. Solution to a geometrical problem in probability. *Ann. Eugenics*, 9 :315–320, 1939.

[86] D. Stoyan, W. S. Kendall, and J. Mecke. *Stochastic geometry and its applications.* John Wiley & Sons Ltd., Chichester, 1987. With a foreword by D. G. Kendall.

[87] D. W. Stroock. *Probability theory, an analytic view.* Cambridge University Press, Cambridge, 1993.

[88] A. S. Sznitman. Long time asymptotics for the shrinking Wiener sausage. *Comm. Pure Appl. Math.*, 43(6) :809–820, 1990.

[89] J. C. Tanner. Polygons formed by random lines in a plane : some further results. *J. Appl. Probab.*, 20(4) :778–787, 1983.

[90] R. van de Weygaert. Fragmenting the Universe III. The construction and statistics of 3-D Voronoi tessellations. *Astron. Astrophys.*, 283 :361–406, 1994.

[91] G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire : Recherches sur les parallélloèdres primitifs. *J. für die reine und angewandte Math.*, 133 :97–178, 1908.

[92] B. Widom. Random sequential addition of hard spheres to a volume. *J. Chem. Phys.*, 44(10) :3888–3894, 1966.

[93] N. Wiener. The ergodic theorem. *Duke Math.*, 5 :1–18, 1939.

[94] S. A. Zuyev. Estimates for distributions of the Voronoĭ polygon's geometric characteristics. *Random Structures Algorithms*, 3(2) :149–162, 1992.